Calculating Scattering Probabilities Using the Chirality-Flow Method

Andrew Lifson

In collaboration with Christian Reuschle & Malin Sjödahl

Work in Progress

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Outline

1. What we Calculate in an Experiment: Cross Sections and Amplitudes
2. Calculating an Amplitude: Feynman Diagrams and the Spinor-Helicity Method
3. Motivating a (Massless) Chirality-Flow Method for Amplitude Calculations + Examples
4. Summary & Outlook

What We Calculate in a Collider Experiment

- In any detector we count events
- $N_{events} = \sigma I$
  - $\sigma =$ Cross section, defined by the type of interaction
  - $I =$ Intensity, parameter of experiment
- Experimentalist's job:
  - Design experiment
  - Measure N_{events} for particular final state, I for machine
- Theorist's job:
  - Calculate/predict cross-section σ for different processes
  - Relate the results to a given underlying theory

Overview: How to Calculate σ in Proton-Proton Collisions

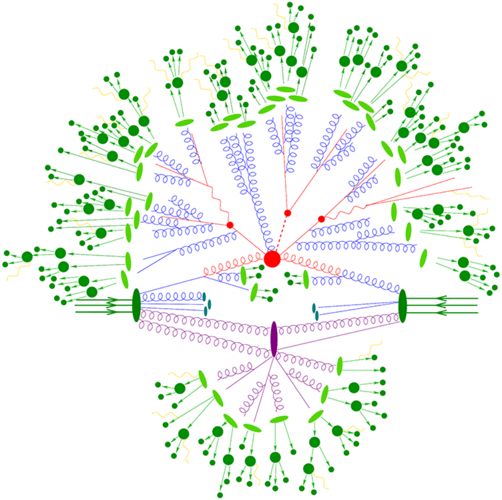

Figure stolen from Stefan Hoeche

- Hard Process, resonant decays
- Parton Shower
- Hadronisation
- PDFs: Pick a parton from a hadron
- Multi-Parton Interactions (MPIs)
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- Hadronic rescattering
- Photon Emission
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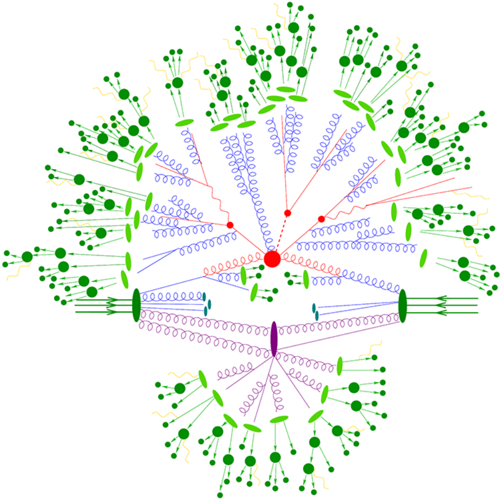

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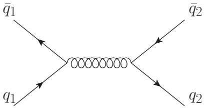
From Cross Sections to Scattering Amplitudes

- Cross section contains cross-section $\hat{\sigma}$ of hard process
- $\hat{\sigma} = \underbrace{\frac{1}{4F}\frac{1}{2\pi^{3n_f-4}}\int\prod_f \frac{d^3p_f}{2E_f}\delta^4(p_{in}-p_{out})}_{\text{kinematics: Easy}}\overline{|M|}^2$
- $\overline{|M|}^2 \equiv$ Squared Scattering Amplitude (spins summed/averaged)
  - Difficult and time consuming to calculate
  - Theory dependent
  - Traditionally use Feynman diagrams to calculate

Calculating an Amplitude: Feynman Diagrams

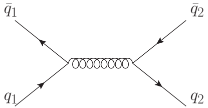
- Feynman diagrams are pictures which encode mathematical objects
- Amplitude $M = \sum$ allowed Feynman diagrams
- e.g.

$$M(q_1\bar{q}_1 \to q_2\bar{q}_2) =$$



$$= t^a_{q_1\bar{q}_1} t^a_{q_2\bar{q}_2} i g_s^2 \frac{\bar{v}(\bar{q}_1)\gamma^\mu u(q_1)\bar{u}(q_2)\gamma_\mu v(\bar{q}_2)}{(p_{q_1}+p_{\bar{q}_1})^2}$$

- Amplitude M is now a complex 4×4 matrix
- Squared amplitude $\overline{|M|}^2 \sim Tr \left(\left| \quad \right|^2 \right)$



Simplifying the Calculation: the Spinor-Helicity Idea

- Can simplify if each external particle has an explicit helicity
  - Helicity is spin projected onto motion
- Feyman diagrams now called Spinor-helicity diagrams
  - Spinor-helicity diagram is a complex number – easy to square
- Lorentz algebra $so(3,1) \cong \underbrace{su(2)}_{\text{left-chiral}} \oplus \underbrace{su(2)}_{\text{right-chiral}}$
- Dirac spinors reducible into two irreps of diff chirality
  - Weyl rep of Dirac algebra naturally separates the two irreps
  - For hard processes, often $p \gg m \Rightarrow m \to 0$, chirality $\sim$ helicity

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- Lorentz algebra $so(3,1) \cong \underbrace{su(2)}_{\text{left-chiral}} \oplus \underbrace{su(2)}_{\text{right-chiral}}$
  - $\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^{\mu,\dot{\alpha}\beta} \\ \sqrt{2}\bar{\tau}^\mu_{\alpha\dot{\beta}} & 0 \end{pmatrix}$, $\sqrt{2}\tau^{\mu,\dot{\alpha}\beta} = \sigma^{\mu,\dot{\alpha}\beta}$
  - $v(p) = \begin{pmatrix} \tilde{\lambda}^{\dot{\alpha}}_p \\ \lambda_{p,\alpha} \end{pmatrix}$, $\bar{u}(p) = \begin{pmatrix} \tilde{\lambda}_{p,\dot{\alpha}} & \lambda^\alpha_p \end{pmatrix}$
  - $\varepsilon^\mu_+(p,r) = \dfrac{\tilde{\lambda}_{p,\dot{\alpha}}\tau^{\mu,\dot{\alpha}\beta}\lambda_{r,\beta}}{\langle rp\rangle}$, $\varepsilon^\mu_-(p,r) = \dfrac{\lambda^\alpha_p\bar{\tau}^\mu_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}_r}{[pr]}$
- Final result in terms of inner products:
  - $\lambda^\alpha_i\lambda_{j\alpha} \equiv \langle ij\rangle$, $\tilde{\lambda}_{i,\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}_j \equiv [ij]$, $\langle ij\rangle, [ij] \sim \sqrt{(p_i+p_j)^2}$
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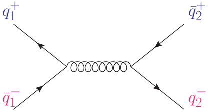
  - e.g. 

$= t^a_{q_1\bar{q}_1} t^a_{q_2\bar{q}_2} \frac{ig_s^2}{(p_{q_1}+p_{\bar{q}_1})^2}[q_1\bar{q}_2]\langle q_2\bar{q}_1\rangle$

Define Problem

- Can we still improve on this?
  - Deriving spinor inner products $\langle ij \rangle$, $[kl]$ requires at least 2 steps
    - Re-write every object as spinors
    - Use Fierz identity $\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tau_{\mu}^{\dot{\alpha}\beta} = \delta_{\alpha}^{\ \beta}\delta^{\dot{\alpha}}_{\ \dot{\beta}}$
  - Not intuitive which inner products we obtain
- In SU(N) use graphical reps for calculations, e.g. Fierz id.
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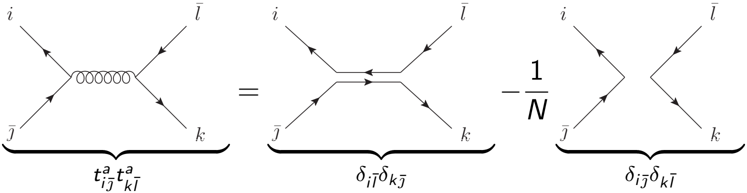

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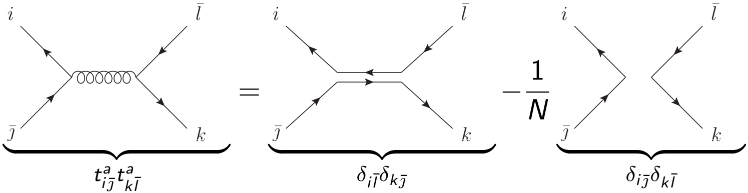

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Creating a Chirality Flow: Defining Fundamental Objects

- Two key differences:
  - Colour $\equiv$ single $SU(N)$: generators $t^a \to \delta$'s
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  - $\lambda_i^{\alpha}\lambda_{j,\alpha} = \langle ij\rangle = -\langle ji\rangle = i \longrightarrow j$
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  - $\lambda_{j,\alpha} = \bullet\!\longrightarrow j$, $\lambda_i^{\alpha} = \bullet\!\longleftarrow i$, $\delta_{\alpha}^{\ \beta} = \alpha \longrightarrow \beta$
  - $\tilde{\lambda}_{i,\dot{\alpha}} = \bullet \dashleftarrow i$, $\tilde{\lambda}_j^{\dot{\alpha}} = \bullet \dashrightarrow j$, $\delta^{\dot{\beta}}_{\ \dot{\alpha}} = \dot{\beta} \dashrightarrow \dot{\alpha}$

Fierz Identity in Chirality Flow

- Fierz required in every helicity-flow calculation
    - Fierz identity with indices: $\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tau^{\dot{\alpha}\beta}_{\mu} = \delta_{\alpha}^{\ \beta}\delta^{\dot{\alpha}}_{\ \dot{\beta}}$
    - Fierz identity with flow:
    - $\Rightarrow \tau^{\mu,\dot{\alpha}\beta} =$, $\bar{\tau}^{\mu}_{\alpha\dot{\beta}} =$
    - $\Rightarrow g_{\mu\nu} =$, or
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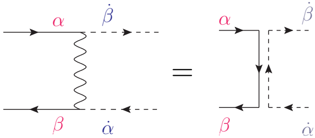
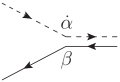
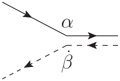


  - $\Rightarrow \tau^{\mu,\dot{\alpha}\beta} =$  , $\bar{\tau}^{\mu}_{\alpha\dot{\beta}} =$ 

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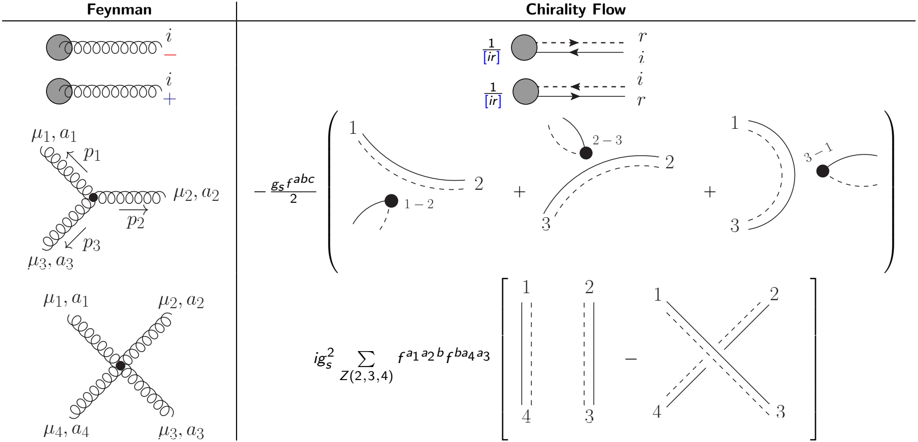
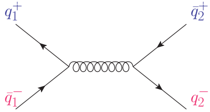
Other Useful Chirality-Flow Rules



The Simplest QCD Example



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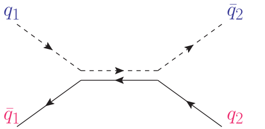

chirality flow

- Immediately read off inner products
- Obvious which inner products we obtained
- Regular spinor-hel requires a few steps
- This is actually a QED-type diagram

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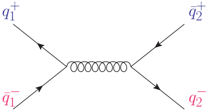

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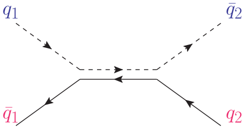

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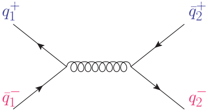

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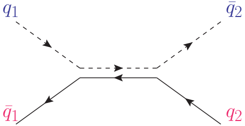

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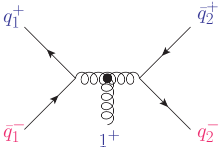
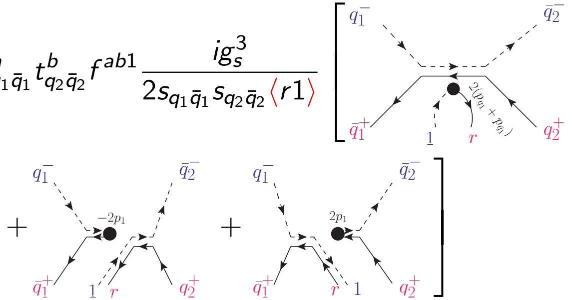
The Simplest Non-Abelian QCD Example



$$= t^a_{q_1\bar{q}_1} t^b_{q_2\bar{q}_2} f^{ab1} \frac{i g_s^3}{2 s_{q_1\bar{q}_1} s_{q_2\bar{q}_2} \langle r1 \rangle}$$



- Few-line calculation written in one step

Summary

- Chirality flow allows for single-line calculation of Feynman diagram
- Also gives transparent/intuitive picture of inner products
- In contrast, spinor-hel method:
  - Requires multiple steps
  - Final result intransparent/unintuitive
- Massless QED and QCD tree-level done
- Useful for any generator using diagrams to avoid dealing with Lorentz algebra

Outlook

- Initial paper coming soon
- Joakim Alnefjord (Master's student) working to complete the SM at tree level
- Loop calculations
- Applications within generator(s)
- Amplitude-level calculations

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- Key difference:
  - Colour $\equiv$ single $SU(N)$: generators $t^a \to \delta$'s
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  - $\not{p} = \sqrt{2} p^\mu \tau_\mu^{\dot{\alpha}\beta} \overset{p^2=0}{=} \tilde{\lambda}_p^{\dot{\alpha}} \lambda_p^\beta =$

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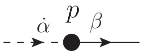
  - $\not{p} = \sqrt{2} p^\mu \tau_\mu^{\dot{\alpha}\beta} \overset{p^2=0}{=} \tilde{\lambda}_p^{\dot{\alpha}} \lambda_p^{\beta} =$ 

Creating a Chirality Flow: Part 2

- Third step: Fermion-vector vertices and vector propagators
  - vertices $\frac{\gamma^\mu}{\sqrt{2}} \to \tau^\mu, \bar{\tau}^\mu$ contracted with vector propagator $g_{\mu\nu}$
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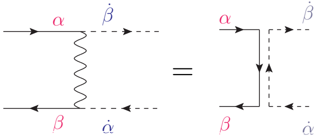

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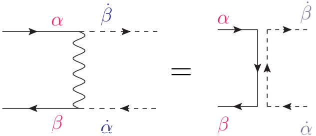
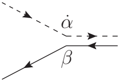
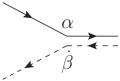
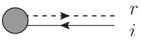
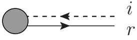
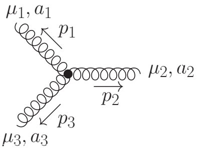
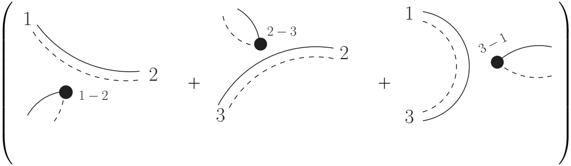
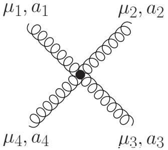
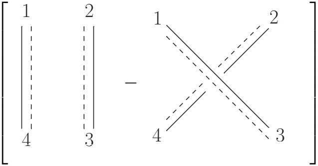


  - $\Rightarrow \tau^{\mu,\dot{\alpha}\beta} =$  , $\bar{\tau}^\mu_{\alpha\dot{\beta}} =$ 

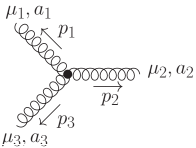
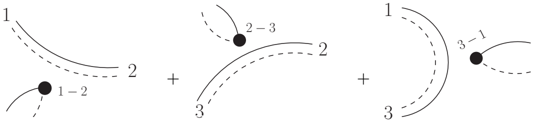
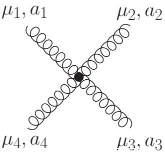
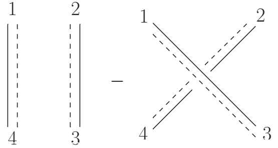
  - $\Rightarrow g_{\mu\nu} =$  , or 
  - Fierz identity already utilised in flow rule

Creating a Chirality Flow: Part 3

- Fourth step: External gauge bosons
  - $\varepsilon_-^\mu(p_i, r) = \frac{1}{[ir]}$  , $\varepsilon_+^\mu(p_i, r) = \frac{1}{\langle ri \rangle}$ 
- Fifth step: Non-abelian vertices

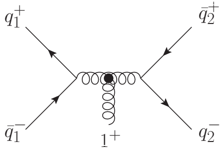
Feynman	Flow
	$-\frac{g_s f^{abc}}{2}$ 
	$ig_s^2 \sum_{Z(2,3,4)} f^{a_1 a_2 b} f^{b a_4 a_3}$ 

The Non-abelian Massless QCD Flow Vertices

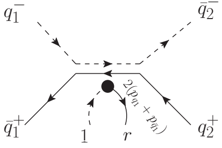
Feynman	**Flow**
	$-\frac{g_s f^{abc}}{2}\Bigg($  $\Bigg)$
	$ig_s^2 \sum_{Z(2,3,4)} f^{a_1 a_2 b} f^{b a_4 a_3} \Bigg[$  $\Bigg]$

QCD Example: $q_1\bar{q}_1 \to q_2\bar{q}_2 g$

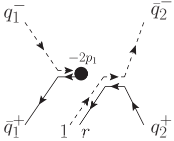
- Triple-gluon vertex provides new structures



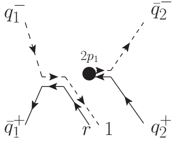
$$= \frac{ig_s^3}{2 s_{q_1\bar{q}_1} s_{q_2\bar{q}_2} \langle r1 \rangle} \Bigg[$$



$+$

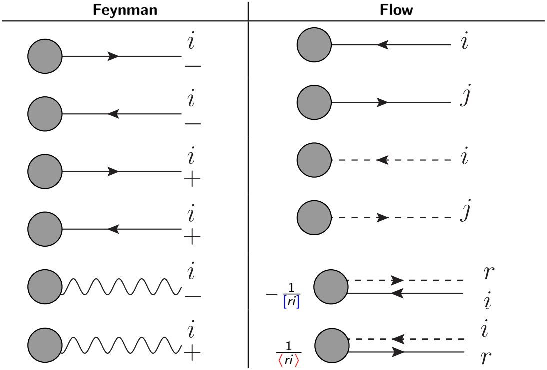


$+$



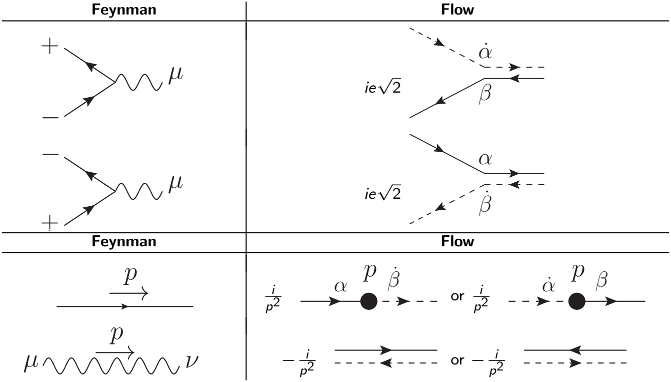
$\Bigg]$

The QED Flow Rules: External Particles



Everything already Fierzed, in terms of spinors

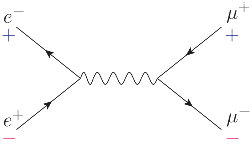
The QED Flow Rules: Vertices and Propagators



Everything already Fierzed, in terms of spinors

Simplest QED Example

- Regular spinor-helicity $\equiv$ easy



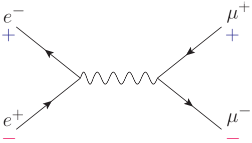
$$= \frac{2ie^2}{s_{e^+e^-}}(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_\mu^{\dot{\alpha}\beta}\lambda_{e^+,\beta})(\lambda_{\mu^-}^{\alpha}\bar{\tau}^\mu_{\alpha\dot{\beta}}\tilde{\lambda}_{\mu^+}^{\dot{\beta}})$$

$$= \frac{2ie^2}{s_{e^+e^-}}\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}_{\mu^+}^{\dot{\alpha}}\lambda_{\mu^-}^{\beta}\lambda_{e^+,\beta} \equiv \frac{2ie^2}{s_{e^+e^-}}[e^-\mu^+]\langle\mu^-e^+\rangle$$

- Helicity flow $\equiv$ super easy and intuitive

Simplest QED Example

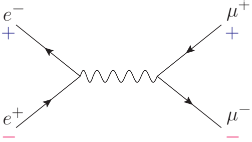
- Regular spinor-helicity $\equiv$ easy



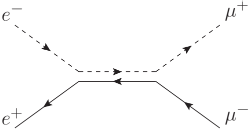
$$= \frac{2ie^2}{s_{e^+e^-}} (\tilde{\lambda}_{e^-,\dot{\alpha}} \tau_{\mu}^{\dot{\alpha}\beta} \lambda_{e^+,\beta})(\lambda_{\mu^-}^{\alpha} \bar{\tau}^{\mu}_{\alpha\dot{\beta}} \tilde{\lambda}_{\mu^+}^{\dot{\beta}})$$

$$= \frac{2ie^2}{s_{e^+e^-}} \tilde{\lambda}_{e^-,\dot{\alpha}} \tilde{\lambda}_{\mu^+}^{\dot{\alpha}} \lambda_{\mu^-}^{\beta} \lambda_{e^+,\beta} \equiv \frac{2ie^2}{s_{e^+e^-}} [e^- \mu^+] \langle \mu^- e^+ \rangle$$

- Helicity flow $\equiv$ super easy and intuitive

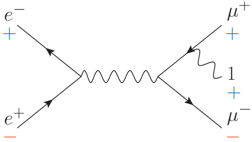


$$= \frac{2ie^2}{s_{e^+e^-}}$$



Next Simplest QED Example

- Regular spinor-helicity $\equiv$ easy



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_{\mu}^{\dot{\alpha}\beta}\lambda_{e^+,\beta}\right)\left(\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}(\not{p}_1+\not{p}_{\mu^+})^{\dot{\beta}\delta}\not{\epsilon}^{+}_{\delta\dot{\gamma}}(1,r)\tilde{\lambda}^{\dot{\gamma}}_{\mu^+}\right)$$

$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_{\mu}^{\dot{\alpha}\beta}\lambda_{e^+,\beta}\right)\tilde{\lambda}_{1,\dot{\delta}}\tilde{\lambda}^{\dot{\delta}}_{\mu^+}$$

$$\times\left(\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}_{1}\lambda^{\delta}_{1}\lambda_{r,\delta}+\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}_{\mu^+}\lambda^{\delta}_{\mu^+}\lambda_{r,\delta}\right)$$

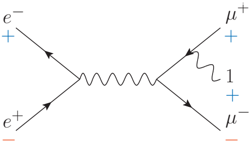
$$\sim \lambda^{\beta}_{\mu^-}\lambda_{e^+,\beta}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_{1}\lambda^{\delta}_{1}\lambda_{r,\delta}+\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_{\mu^+}\lambda^{\delta}_{\mu^+}\lambda_{r,\delta}\right)\tilde{\lambda}_{1,\dot{\delta}}\tilde{\lambda}^{\dot{\delta}}_{\mu^+}$$

Correct Answer

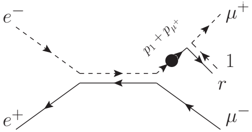
$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\Big([e^-1]\langle 1r\rangle+[e^-\mu^+]\langle\mu^+r\rangle\Big)[1\mu^+]\langle\mu^-e^+\rangle$$

Next Simplest QED Example

- Helicity flow $\equiv$ super easy and intuitive



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}\,s_{\mu^+1}\langle r1\rangle}$$



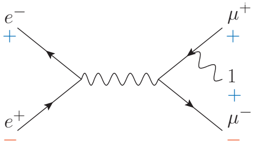
- Immediately read off inner products

Correct Answer

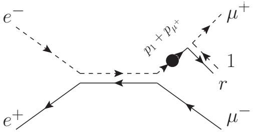
$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}\,s_{\mu^+1}\langle r1\rangle}\Big([e^-1]\langle 1r\rangle + [e^-\mu^+]\langle \mu^+ r\rangle\Big)[1\mu^+]\langle \mu^- e^+\rangle$$

Next Simplest QED Example

- Helicity flow $\equiv$ super easy and intuitive



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}$$



- Immediately read off inner products

Correct Answer

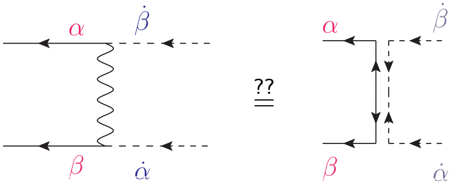
$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\Big([e^-1]\langle 1r\rangle + [e^-\mu^+]\langle \mu^+ r\rangle\Big)[1\mu^+]\langle \mu^- e^+\rangle$$

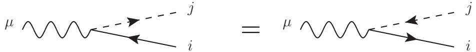
Calculating A_i: the Spinor-Helicity Method

- Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
- Weyl representation of Dirac algebra naturally separates the two reps
  - $\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^{\mu,\dot{\alpha}\beta} \\ \sqrt{2}\bar{\tau}^\mu_{\alpha\dot{\beta}} & 0 \end{pmatrix}$, $\sqrt{2}\tau^\mu = (1, \vec{\sigma})$, $\sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma})$
  - $\mathrm{Tr}(\tau^\mu \bar{\tau}^\mu) = g^{\mu\nu}$, $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$
  - $u(p) = \begin{pmatrix} u_-(p) \\ u_+(p) \end{pmatrix} = \begin{pmatrix} v_+(p) \\ v_-(p) \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}^{\dot{\alpha}}_p \\ \lambda_{p,\alpha} \end{pmatrix}$, $\bar{u}(p) = \begin{pmatrix} \tilde{\lambda}_{p,\dot{\alpha}} & \lambda^\alpha_p \end{pmatrix}$
- Final result in terms of inner products:
  - $\lambda^\alpha_i \lambda_{j\alpha} \equiv \langle ij \rangle$, $\tilde{\lambda}_{i,\dot{\beta}} \tilde{\lambda}^{\dot{\beta}}_j \equiv [ij]$, $\langle ij \rangle, [ij] \sim \sqrt{s_{ij}}$
- $\varepsilon^\mu_+(p,r) = \dfrac{\tilde{\lambda}_{p,\dot{\alpha}} \tau^{\mu,\dot{\alpha}\beta} \lambda_{r,\beta}}{\langle rp \rangle}$, $\varepsilon^\mu_-(p,r) = \dfrac{\lambda^\alpha_p \bar{\tau}^\mu_{\alpha\dot{\beta}} \tilde{\lambda}^{\dot{\beta}}_r}{[pr]}$
- No complicated traces of γ matrices, rather simple identities like:
  - $(\tilde{\lambda}_{i,\dot{\alpha}} \tau^{\dot{\alpha}\beta}_\mu \lambda_{j,\beta})(\lambda^\gamma_k \bar{\tau}^\mu_{\gamma\dot{\delta}} \tilde{\lambda}^{\dot{\delta}}_l) = \lambda^\beta_i \lambda_{k\beta} \tilde{\lambda}_{l,\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}}_j = \langle ik \rangle [lj]$

Fun with Arrows and the Fierz Identity

- Sometimes have to contract $\tau^{\mu}\tau_{\mu}$ or $\bar{\tau}^{\mu}\bar{\tau}_{\mu}$
- This would lead to arrows pointing towards each other, e.g.



- To fix, use charge conservation of a current:
  - $\lambda_i^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}_j^{\dot{\beta}} = \tilde{\lambda}_{j,\dot{\alpha}}\tau^{\mu,\dot{\alpha}\beta}\lambda_{i,\beta}$
- Or in pictures:
  - 

How to Calculate a (Massless) Scattering Amplitude

- QCD often factorise colour, use helicity basis for kinematics

$$\mathcal{M}_h\left(1^{h_1},\dots,n^{h_n}\right)=\sum_i C_i A_i\left(p_1^{h_1},\dots,p_n^{h_n}\right)$$

- $C_i \equiv$ colour factor
  - QED: $C_i = 1$
- $A_i \equiv$ kinematic amplitude
  - Cross incoming particles to outgoing
  - Each particle j is given a specific helicity h_j
  - Since massless, helicity $\sim$ chirality