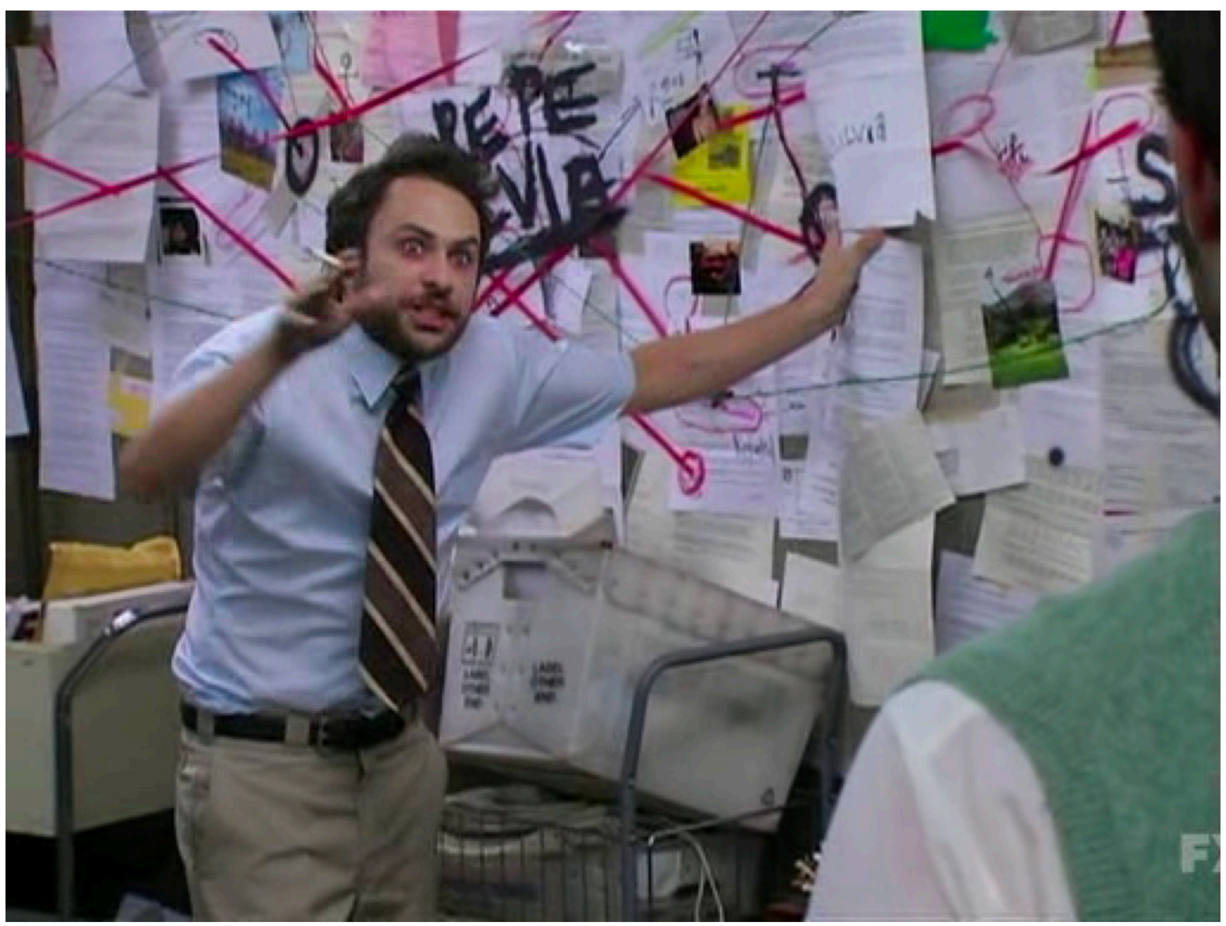
# Local Unitarity

A representation of differential cross-sections that is locally free of IR singularities at any order

LU: Zeno Capatti, in collaboration with V. Hirschi, A. Pelloni and B. Ruijl
LTD, cLTD and Contour Deformation: ZC, in collaboration with V. Hirschi, D. Kermanschah, A. Pelloni and B. Ruijl

First, allow me to point out a striking observation, might be inspiration for new physics…

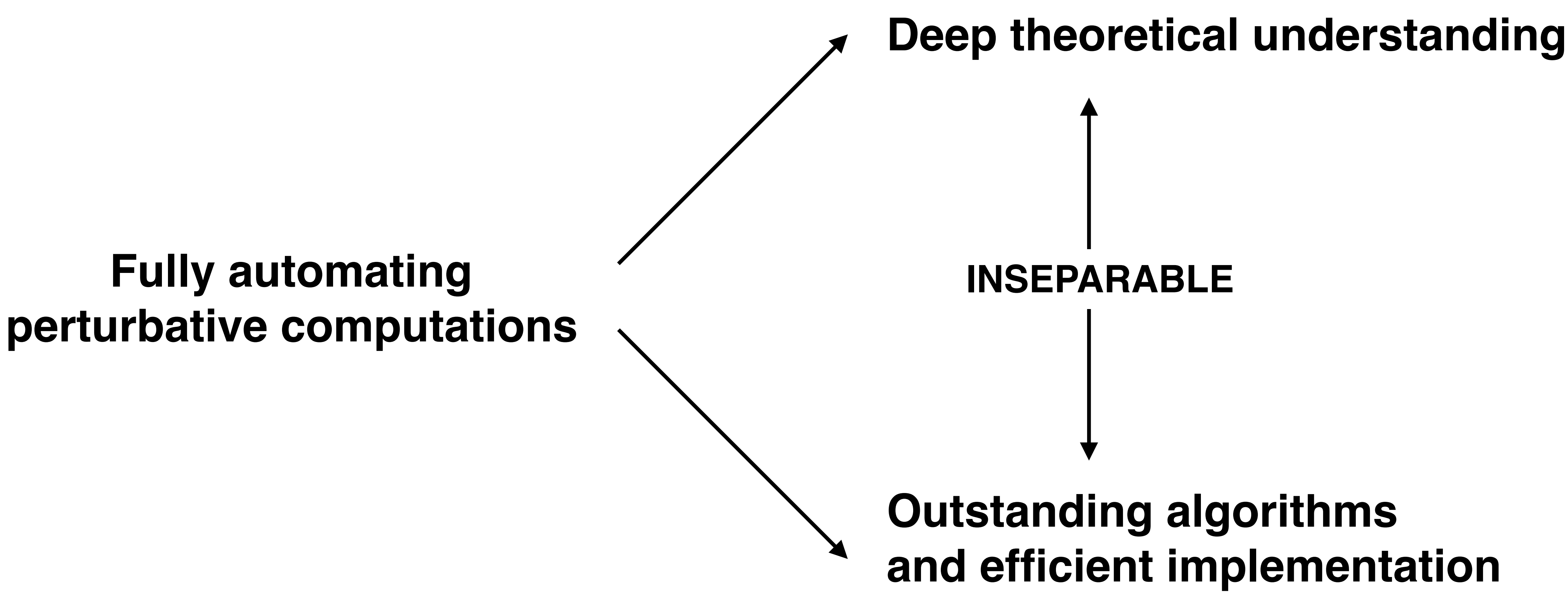
**Lund University** → **LU** ← **Local Unitarity**



This cannot be a coincidence!!

# Our objective

**Automating perturbative computations**: provide a deterministic procedure (and code) that, given any process specific input, and given enough time and enough computational resources, outputs a reliable output with arbitrary precision



The hardest theoretical problem in full automation is that of **IR singularities**.

It manifests itself in fixed order computations, PDFs, event generators.

↓

**LU forces to unify the treatment for all of its manifestations!**

In contrast, the traditional way of computing cross-section usually divides the problem into

- **Computing amplitudes analytically**
- **Computing the phase space integrals numerically with counter-terms**

This asymmetric way of dealing with IR singularities **hides an inherent simplicity**

Some relevant pragmatic consequences…

- **No counter-terms**
- **No dimensional regularisation**
- **Not process specific**
- **Fully numerical and automatable**
- **Differential**

Some interesting theoretical consequences…

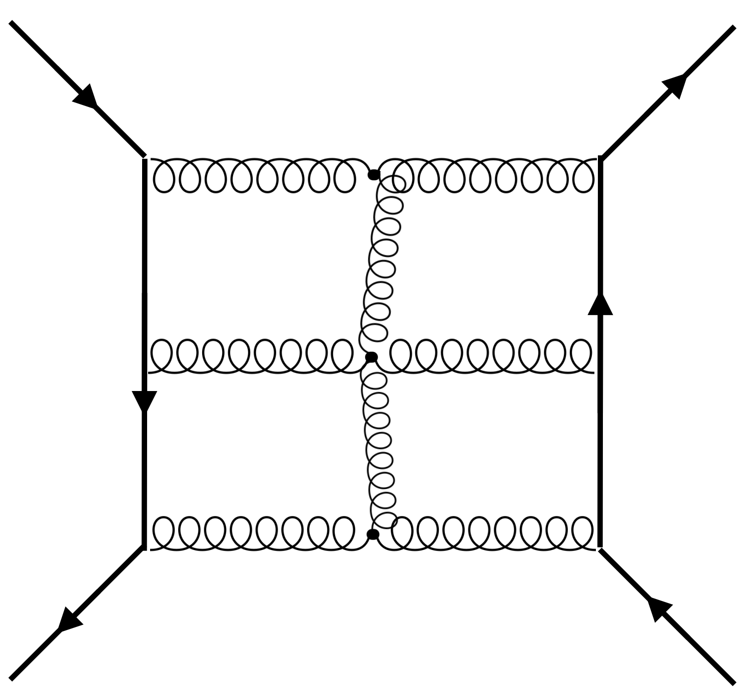
- **Forward scattering diagrams are central, not amplitudes**
- **Initial states with higher multiplicities**
- **Beyond LSZ**
- **Infrared scales from theory**
- **Classification of singularities and the systematics of their cancellations**
- **No explicit reference to collinear mass factorisation**

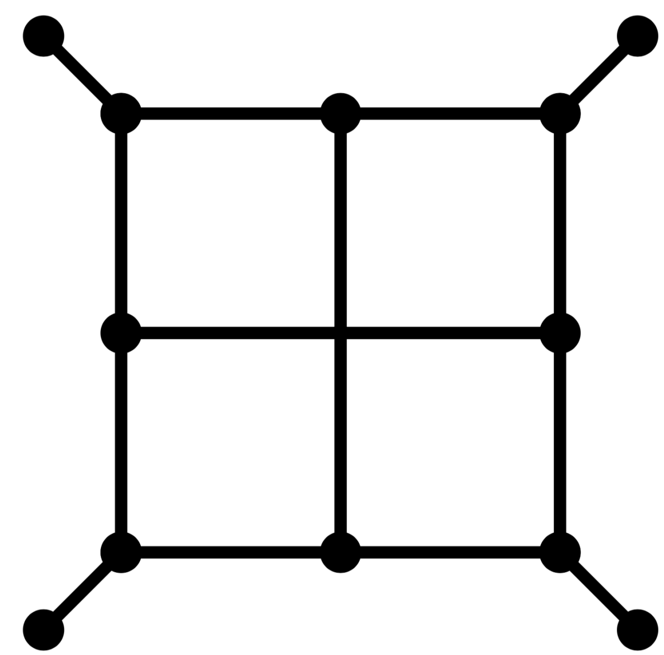
In computing perturbative cross section for physical processes in QFTs, one encounters diagrams, either in the form of amplitudes or forward scattering diagrams

**The properties of perturbative cross-sections are deeply entrenched with the diagrammatic technique**

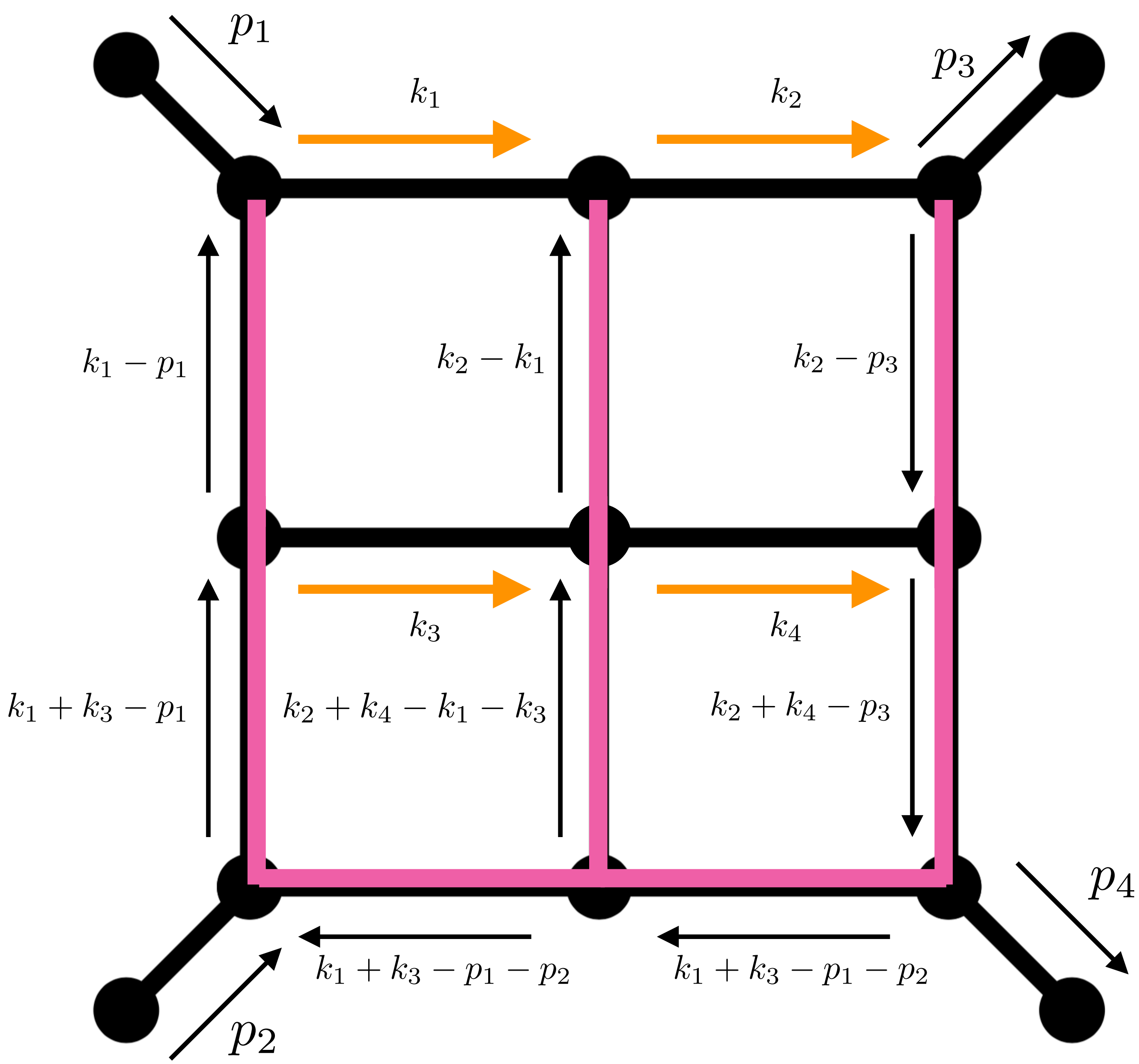
As a recurring example, one can consider a four-loop amplitude for $q\bar{q} \to q\bar{q}$

Or a forward-scattering diagram for N4LO $q\bar{q} \to X$





# Momentum conservation constraints




Choosing a **loop momentum** routing is equivalent to fixing a **spanning tree**

The edges not in the spanning tree are the loop variables!

Spanning trees contain info on the **connectivity structure** of the graph

*Indeed a graph admits a spanning tree only if it is connected!*

**Momentum conservation completely determines the singular structure!!!**

# Loop-Tree Duality

Consider a loop integral in the Minkowski representation

$$I = \int \left( \prod_{i=1}^{L} \frac{d^4 k}{(2\pi)^4} \right) \frac{N}{\prod_{j\in\mathbf{e}} (q_j^2 - m_j^2)}$$

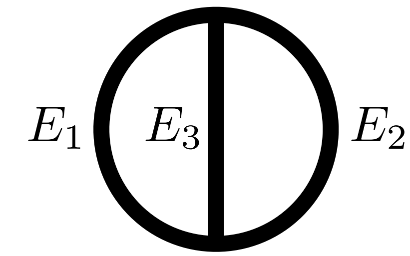
The **LTD representation** using residue theorem to integrate the energy components

$$I = (-\mathrm{i})^L \int \left( \prod_{i=1}^{L} \frac{d^3 \vec{k}}{(2\pi)^3} \right) f_{\mathrm{ltd}}$$

Our objective is to determine $f_{\mathrm{ltd}}$ for any Feynman diagram

**The interplay between momentum conservation conditions and the $\mathrm{i}\epsilon$ prescription is key in deriving $f_{\mathrm{ltd}}$**

Choose the simplest non trivial example: the **two-loop sunrise**




$$f_{\rm ltd} = -\int dk_1^0 dk_2^0 \frac{N}{(k_1^0+E_1)(k_1^0-E_1)(k_2^0+E_2)(k_2^0-E_2)(k_1^0+k_2^0+E_3)(k_1^0+k_2^0-E_3)}$$

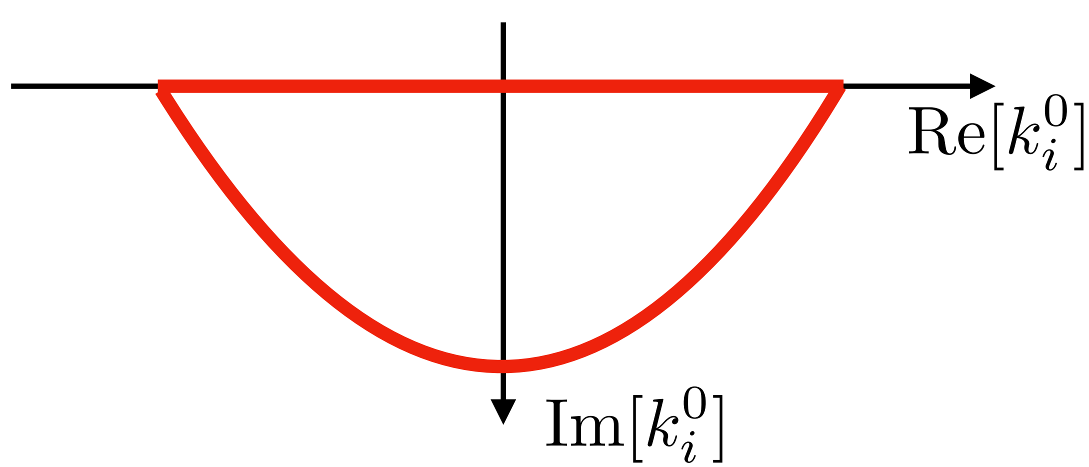
$$k_1^2 - m_1^2 + i\epsilon = (k_1^0 - E_1)(k_1^0 + E_1)$$

where

$$E_1 = \sqrt{|\vec{k}_1|^2 + m_1^2 - \mathrm{i}\epsilon}, \quad E_2 = \sqrt{|\vec{k}_2|^2 + m_2^2 - \mathrm{i}\epsilon}, \quad E_3 = \sqrt{|\vec{k}_1 + \vec{k}_2|^2 + m_3^2 - \mathrm{i}\epsilon}$$

Due to the **Feynman prescription** $\mathrm{Im}[E_i] < 0$

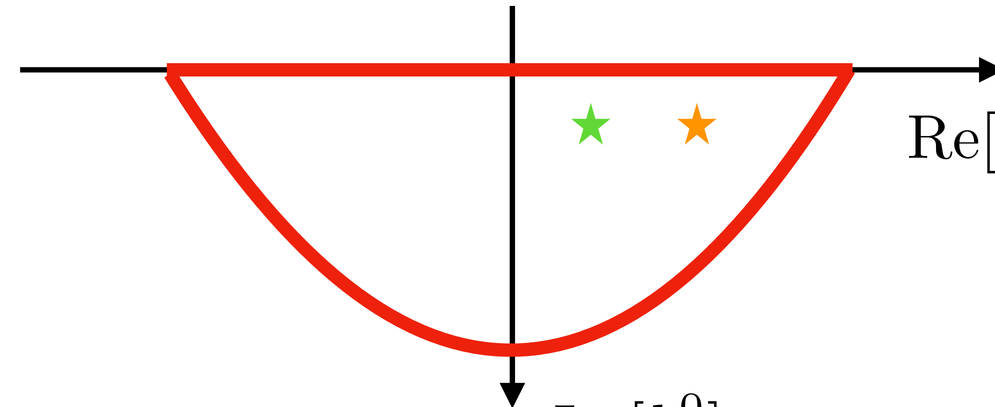
**Analytically continue** the integrand in $k_1^0$ **first**, and **then** $k_2^0$. We choose to close the contour in the lower half of complex plane



Start by performing the integration in $k_1^0$. The poles contained in the contour are

$$\tilde{k}_1^0 = E_1$$

$$\tilde{k}_1^0 = -k_2^0 + E_3$$




Indeed

$$\mathrm{Im}[E_1] < 0$$

$$\mathrm{Im}[-k_2^0 + E_3] = \mathrm{Im}[E_3] < 0$$

Using residue theorem we obtain two residue

$$f_{\mathrm{ltd}} = 2\pi \mathrm{i} \int dk_2^0 \frac{N(k_1^0 = E_1)}{2E_1(k_2^0 + E_2)(k_2^0 - E_2)(k_2^0 + E_1 + E_3)(k_2^0 + E_1 - E_3)}$$

$$+ 2\pi \mathrm{i} \int dk_2^0 \frac{N(k_1^0 = -k_2^0 + E_3)}{2E_3(k_2^0 + E_2)(k_2^0 - E_2)(-k_2^0 - E_1 + E_3)(-k_2^0 + E_1 + E_3)}$$

We now perform the integration in $k_2^0$

$$I_1 = 2\pi\mathrm{i}\int dk_2^0 \frac{N(k_1^0=E_1)}{2E_1(k_2^0+E_2)(k_2^0-E_2)(k_2^0+E_1+E_3)(k_2^0+E_1-E_3)}$$

The poles of this piece are located at

$$\tilde{k}_2^0 = E_2$$
$$\tilde{k}_2^0 = -E_1 + E_3$$

The first pole is always in the lower half of complex plane, the **second is not**!!!

$\mathrm{Im}[-E_1+E_3]$ **Does not have a well-defined negative sign!**

Thus, after applying residue theorem, we write

$$I_1 = \frac{N(k_1^0=E_1, k_2^0=E_2)}{2E_1 2E_2(E_2+E_1+E_3)(E_2+E_1-E_3)}$$
$$+\frac{N(k_1^0=E_1, k_2^0=E_3-E_1)}{2E_1 2E_2(E_2-E_1+E_3)(-E_2-E_1+E_3)}\Theta(\mathrm{Im}[-E_3+E_1])$$

**Ensuring the pole is in the contour!**

$$I_2 = 2\pi \mathrm{i} \int dk_2^0 \frac{N(k_1^0 = -k_2^0 + E_3)}{2E_3(k_2^0 + E_2)(k_2^0 - E_2)(-k_2^0 - E_1 + E_3)(-k_2^0 + E_1 + E_3)}$$

The poles of this piece are located at

$$\tilde{k}_2^0 = E_2$$

$$\tilde{k}_2^0 = E_3 - E_1$$

$$\tilde{k}_2^0 = E_1 + E_3$$

The **second pole** can be inside or outside the contour depending on $E_1,\ E_3$.

After applying residue theorem

$$I_2 = \frac{N(k_1^0 = -E_2 + E_3, k_2^0 = E_2)}{2E_3 2E_2(-E_2 - E_1 + E_3)(-E_2 + E_1 + E_3)}$$
$$- \frac{N(k_1^0 = E_1, k_2^0 = E_3 - E_1)}{2E_3 2E_1(E_3 - E_1 + E_2)(E_3 - E_1 - E_2)} \Theta(\mathrm{Im}[-E_3 + E_1])$$
$$+ \frac{N(k_1^0 = -E_1, k_2^0 = E_1 + E_3)}{2E_3 2E_1(E_1 + E_2 + E_3)(E_1 + E_3 - E_2)}$$

Finally, we can combine the two contributions!

$$f_{\text{ltd}} = I_1 + I_2 = \frac{N(k_1^0 = E_1, k_2^0 = E_2)}{2E_1 2E_2(E_2 + E_1 + E_3)(E_2 + E_1 - E_3)}$$

$$+ \frac{N(k_1^0 = E_1, k_2^0 = E_3 - E_1)}{2E_1 2E_2(E_2 - E_1 + E_3)(-E_2 - E_1 + E_3)} \Theta(\text{Im}[-E_3 + E_1])$$

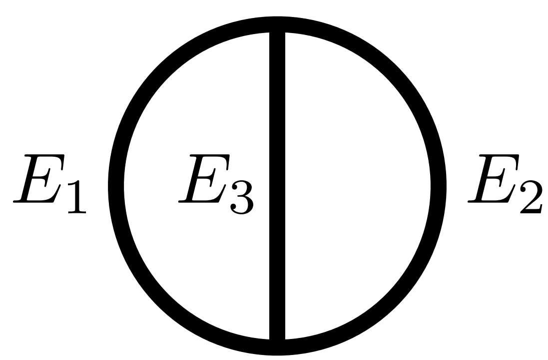
$$+ \frac{N(k_1^0 = -E_2 + E_3, k_2^0 = E_2)}{2E_3 2E_2(-E_2 - E_1 + E_3)(-E_2 + E_1 + E_3)}$$

$$- \frac{N(k_1^0 = E_1, k_2^0 = E_3 - E_1)}{2E_3 2E_1(E_3 - E_1 + E_2)(E_3 - E_1 - E_2)} \Theta(\text{Im}[-E_3 + E_1])$$

$$+ \frac{N(k_1^0 = -E_1, k_2^0 = E_1 + E_3)}{2E_3 2E_1(E_1 + E_2 + E_3)(E_1 + E_3 - E_2)}$$

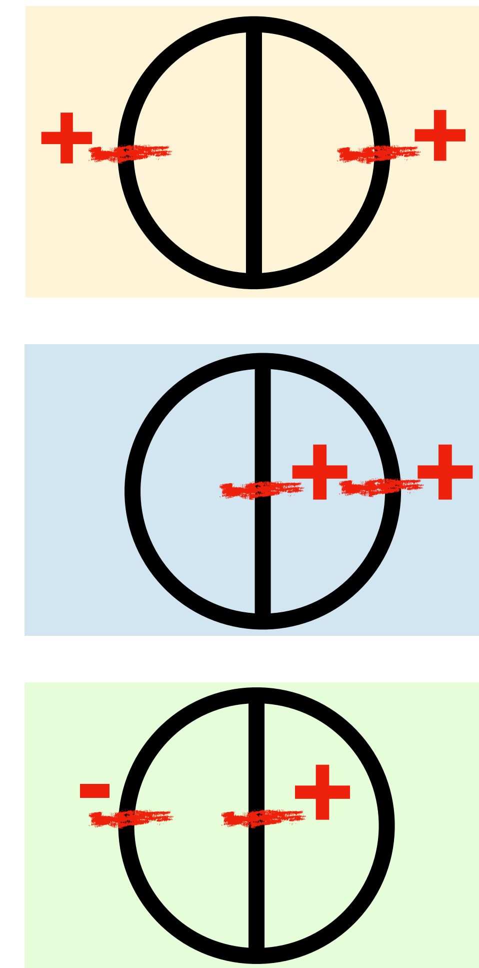
**Same contribution, opposite sign! Theta cancellation!**

We can represent the final result graphically. Using the convention



$$f_{\text{ltd}} = I_1 + I_2 = \frac{N(k_1^0 = E_1, k_2^0 = E_2)}{2E_1 2E_2 (E_2 + E_1 + E_3)(E_2 + E_1 - E_3)} + \frac{N(k_1^0 = -E_2 + E_3, k_2^0 = E_2)}{2E_3 2E_2 (-E_2 - E_1 + E_3)(-E_2 + E_1 + E_3)} + \frac{N(k_1^0 = -E_1, k_2^0 = E_1 + E_3)}{2E_3 2E_1 (E_1 + E_2 + E_3)(E_1 + E_3 - E_2)}$$

**_cut structure_**



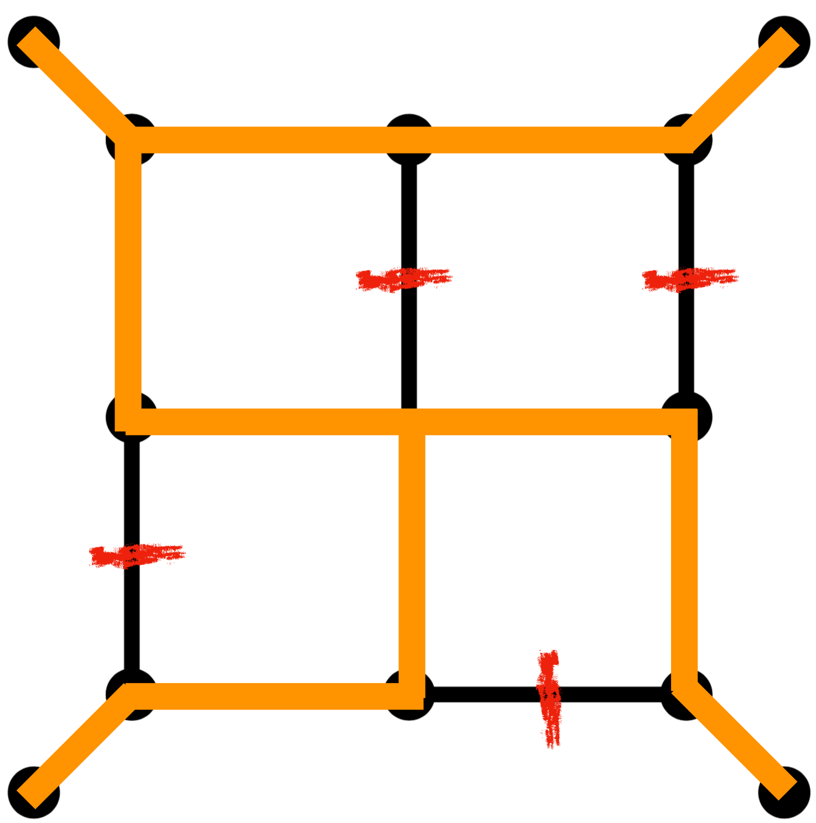
For a general amplitude

$$f_{\text{ltd}} = \sum_{\mathbf{b} \in \mathcal{B}} \int \left( \prod_{i=1}^{L} d^3 \vec{k}_i \right) N \frac{\prod_{j \in \mathbf{b}} \delta^{(\sigma_j^{\mathbf{b}})}(q_j^2 - m_j^2)}{\prod_{i \in \mathbf{e} \setminus \mathbf{b}} (q_i^2 - m_i^2)}$$

Each delta corresponds to a cut with an associated sign or energy flow.

# Real and virtual particles?

LTD clarifies the distinction between **real and virtual particles**



A cut-structure corresponds to a **unique spanning tree**

**Cut particles are "physical"**, i.e. on-shell.

Then the cut structure represents a **classical tree process**

**LTD sums over all possible classical tree processes that can be embedded in the virtual loops**

# Singularities of the sunrise

$$f_{\text{ltd}} = \frac{N(k_1^0 = E_1, k_2^0 = E_2)}{2E_1 2E_2 (E_2 + E_1 + E_3)(E_2 + E_1 - E_3)} + \frac{N(k_1^0 = -E_2 + E_3, k_2^0 = E_2)}{2E_3 2E_2 (-E_2 - E_1 + E_3)(-E_2 + E_1 + E_3)} + \frac{N(k_1^0 = -E_1, k_2^0 = E_1 + E_3)}{2E_3 2E_1 (E_1 + E_2 + E_3)(E_1 + E_3 - E_2)}$$

Let's look at the denominators

$$E_1 = 0, \;\; E_2 = 0, \;\; E_3 = 0$$

$$E_1 + E_2 + E_3 = 0$$

$$E_1 + E_3 - E_2 = 0$$

$$E_3 - E_2 - E_1 = 0$$

However, the last two singularities are singularities of single residues, but not of $f_{\text{ltd}}$ !!!

Using the identity $$\frac{1}{(x+y)(x-y)} = \frac{1}{2y}\left(\frac{1}{x-y} - \frac{1}{x+y}\right)$$

we can rewrite $$f_{\text{ltd}} = \frac{N'}{2E_1 2E_2 2E_3 (E_1 + E_2 + E_3)}$$ **Manifestly Causal LTD**

where $N'$ is a polynomial. This is the phenomenon of **dual cancellations**.

# E-surfaces or physical thresholds

This can be generalised to **any arbitrary amplitude**. The general amplitude will be singular at zeros of on-shell energies and at the locations

$$\eta = \sum_i E_i - p_0 = 0$$

where $p_0$ is a linear combination of the energies of external particles.

$\eta$ **is a positive linear combination of the on-shell energies of internal particles**.

As a consequence, it describes a **convex bounded surface**

Its imaginary part has a well-defined sign

$$\mathrm{Im}\left[\sum_i E_i - p_0\right] = \sum_i \mathrm{Im}[E_i] < 0$$

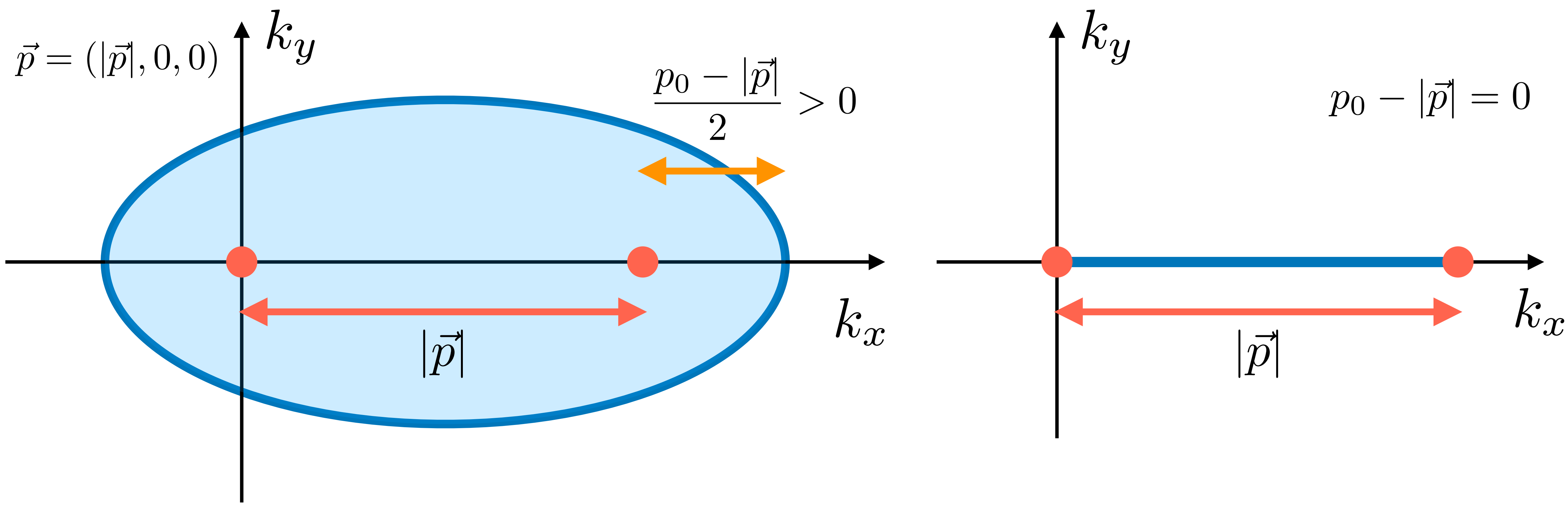
This is important to determine the constraints on the contour deformation

# Ellipses and pinched singularities

At one loop, the physical thresholds take an especially simple form

$$E_1 + E_2 - p_0 = \sqrt{|\vec{k}|^2} + \sqrt{|\vec{k} + \vec{p}|^2} - p_0 = 0$$

It's the equation for an ellipse! It exists if $p^2 \geq 0$

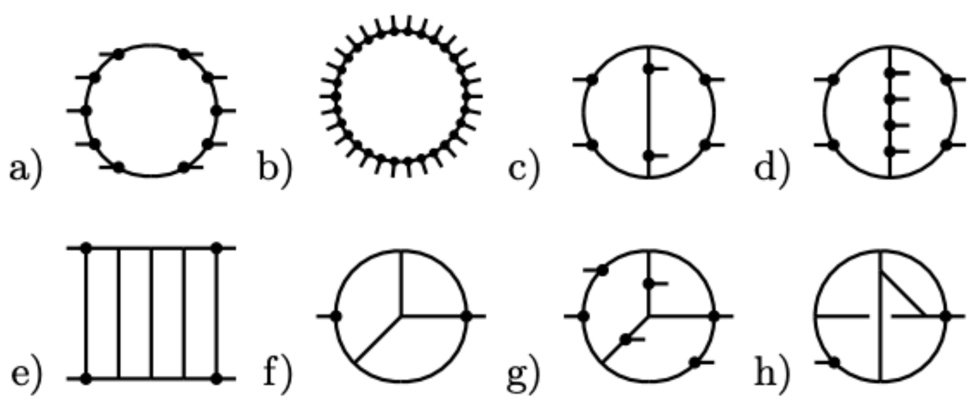



Pinched configuration is obtained by squeezing the ellipse

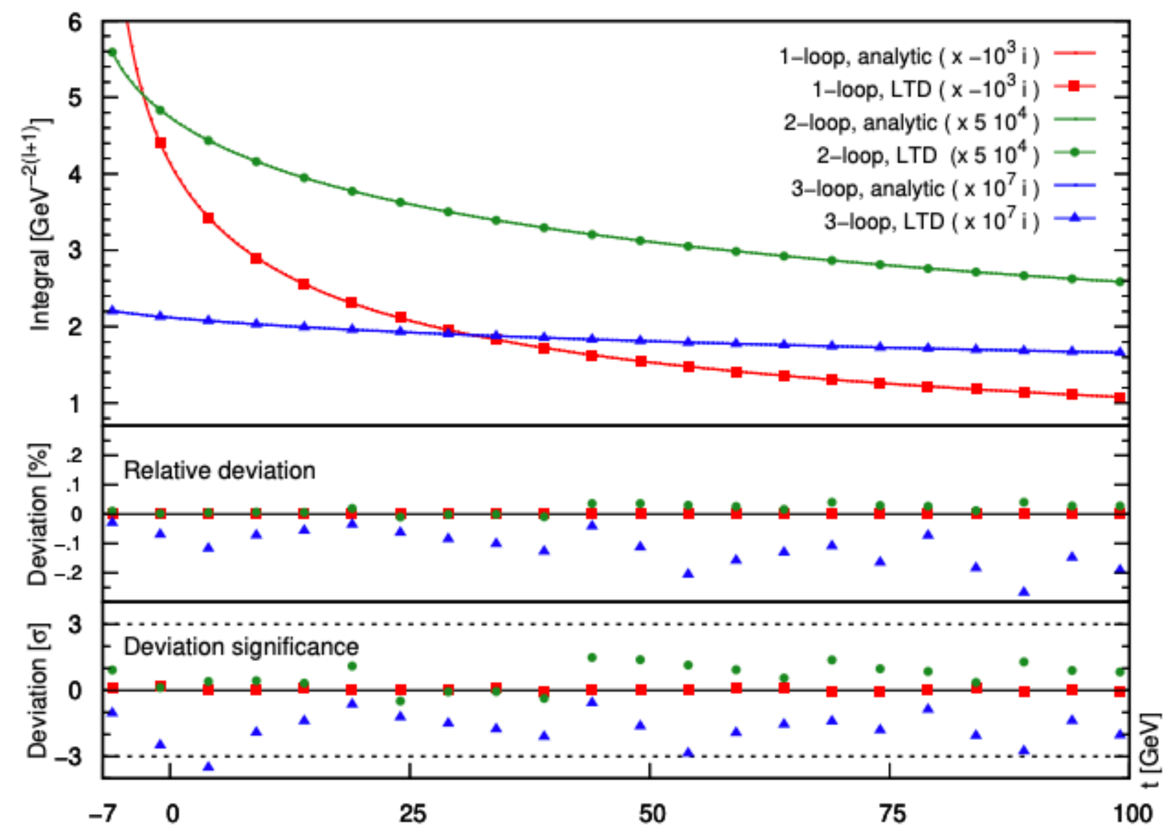
# Numerically computing amplitudes within LTD

In the unphysical region $p_i^2 < 0, \quad \left(\sum_j p_j\right)^2 < 0$ **there are no physical thresholds!**

The integrand is finite and can be easily Monte Carlo integrated



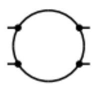
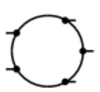
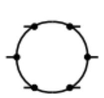

| $G$ | Reference | | Numerical LTD | | N $[10^6]$ | $[\mu s]$ |
|---|---|---|---|---|---|---|
| a)* | [33] | $i\,4.31638$ | $\cdot 10^{-7}$ | $i\,4.31637(19)$ $\cdot 10^{-7}$ | 110 | 1.1 |
| b) | [33] | $i\,0.358640$ | | $i\,0.358646(29)$ | 210 | 5.9 |
| c) | [7] | $1.1339(5)$ | $\cdot 10^{-4}$ | $1.133719(58) \cdot 10^{-4}$ | 5500 | 2.5 |
| c)* | [7] | $4.398(1)$ | $\cdot 10^{-8}$ | $4.39825(17)$ $\cdot 10^{-8}$ | 5500 | 2.5 |
| d)* | [7] | $2.409(1)$ | $\cdot 10^{-8}$ | $2.40869(27)$ $\cdot 10^{-8}$ | 5500 | 3.5 |
| e) | [34] | $-1.433521$ | $\cdot 10^{-6}$ | $-1.4338(18)$ $\cdot 10^{-6}$ | 1500 | 27.4 |
| f) | [35] | $i\,5.26647$ | $\cdot 10^{-6}$ | $i\,5.236(38)$ $\cdot 10^{-6}$ | 7000 | 3.3 |
| g)* | [7] | $i\,1.7790(6)$ | $\cdot 10^{-10}$ | $i\,1.77648(48)$ $\cdot 10^{-10}$ | 22000 | 11 |
| h) | [35] | $-8.36515$ | $\cdot 10^{-8}$ | $-8.309(31)$ $\cdot 10^{-8}$ | 7000 | 15.8 |

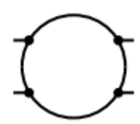
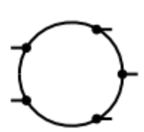


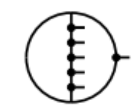
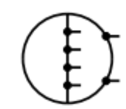
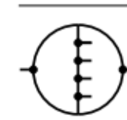
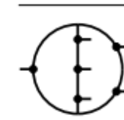
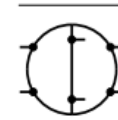
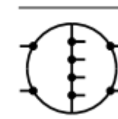
## In the physical region we need a deformation satisfying the causal constraint

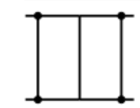
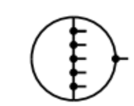
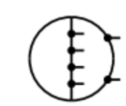
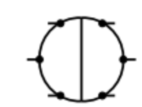
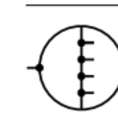
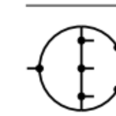
$$\vec{k} \to \vec{k} - \mathrm{i}\kappa \quad \text{with} \quad \kappa \cdot \nabla \eta_i > 0 \text{ if } \eta_i = 0$$

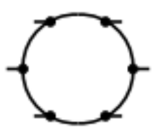
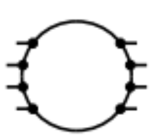
(plus some magnitude constraints)

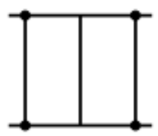
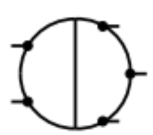
| Topology | Numerical LTD |
|---|---|
|  Box4E | -6.57637 +/- 0.00122 |
| | -7.43805 +/- 0.00121 |
|  1L5P | -3.44317 +/- 0.00045 |
| | -2.56505 +/- 0.00046 |
| | -0.00036 +/- 0.00029 |
| | 5.97143 +/- 0.00029 |
| | -0.83888 +/- 0.00016 |
| | -1.71325 +/- 0.00017 |
| | -3.49044 +/- 0.00054 |
| | -3.89965 +/- 0.00054 |
| | 0.90036 +/- 0.00076 |
| | 4.17823 +/- 0.00080 |
| | 0.04227 +/- 0.00068 |
| | -2.18118 +/- 0.00068 |
|  1L6P | 0.03046 +/- 0.00006 |
| | -1.17691 +/- 0.00008 |
| | -2.07392 +/- 0.00188 |
| | 0.42593 +/- 0.00161 |
| | 1.36950 +/- 0.00052 |
| | -2.25957 +/- 0.00053 |
| | 1.29802 +/- 0.00038 |
| | -2.16555 +/- 0.00037 |
| | -0.27225 +/- 0.00010 |
| | -1.20895 +/- 0.00011 |
| | 2.83777 +/- 0.00040 |
| | 0.83144 +/- 0.00040 |
| | -3.01976 +/- 0.00040 |
| | -7.73280 +/- 0.00047 |
| | 2.13487 +/- 0.03230 |
| | 0.65770 +/- 0.03145 |
| | 0.00804 +/- 0.00014 |
| | -1.15278 +/- 0.00014 |
| | -2.81583 +/- 0.00060 |
| | 2.47308 +/- 0.00061 |

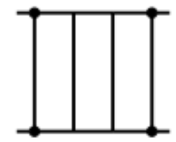
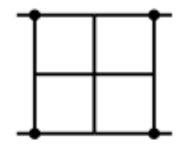
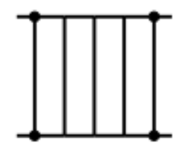
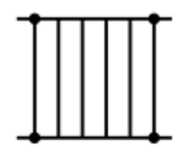
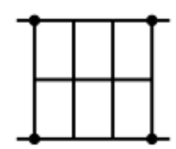
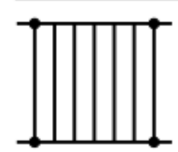
| Topology | Numerical LTD |
|---|---|
|  1L4P | 1.13123 +/- 0.00006 |
| | -0.55486 +/- 0.00005 |
| | 5.71929 +/- 0.00055 |
| | -7.24055 +/- 0.00053 |
| | 1.55376 +/- 0.00012 |
| | -2.07005 +/- 0.00012 |
| | 1.85214 +/- 0.00012 |
| | -2.18397 +/- 0.00012 |
| | 0.30272 +/- 0.00004 |
| | -1.08130 +/- 0.00004 |
| | -0.17991 +/- 0.00005 |
| | -2.27593 +/- 0.00008 |
|  1L5P | -1.90856 +/- 0.00074 |
| | -6.45306 +/- 0.00077 |
| | -0.15137 +/- 0.00032 |
| | -1.80672 +/- 0.00033 |
| | -0.66271 +/- 0.00032 |
| | -1.23567 +/- 0.00032 |
| | 2.60394 +/- 0.00072 |
| | -7.95017 +/- 0.00076 |
| | -0.48305 +/- 0.00059 |
| | -3.27664 +/- 0.00061 |
| | -1.21508 +/- 0.00020 |
| | -1.53126 +/- 0.00020 |

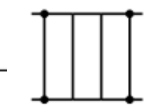
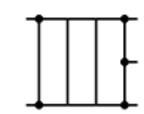
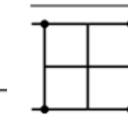
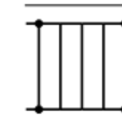
| Topology | Numerical LTD |
|---|---|
|  2L6P.a | 4.58688 +/- 0.05132 |
| | 5.04144 +/- 0.05075 |
| | -1.04316 +/- 0.35247 |
| | -4.42468 +/- 0.35421 |
|  2L6P.b | 1.17336 +/- 0.00888 |
| | 3.99809 +/- 0.00896 |
| | 5.35217 +/- 0.00153 |
| | 3.81579 +/- 0.00150 |
|  2L6P.c | 4.90974 +/- 0.01407 |
| | -2.13974 +/- 0.01434 |
| | 1.05934 +/- 0.15850 |
| | 1.03698 +/- 0.15312 |
|  2L6P.d | 1.90487 +/- 0.05753 |
| | -3.55267 +/- 0.05746 |
| | -2.97419 +/- 0.00961 |
| | -2.18847 +/- 0.00957 |
|  2L6P.e | 2.87833 +/- 0.00951 |
| | 1.99937 +/- 0.00961 |
| | 1.67332 +/- 0.00578 |
| | -0.21788 +/- 0.00571 |
|  2L6P.f | -0.95486 +/- 0.00890 |
| | 3.28530 +/- 0.00889 |
| | 2.55104 +/- 0.00208 |
| | -1.63019 +/- 0.00205 |
|  2L8P | -5.15438 +/- 0.03310 |
| | 6.78546 +/- 0.03243 |

| Topology | Numerical LTD |
|---|---|
|  2L4P.a | 3.82875 +/- 0.00015 |
| | -4.66843 +/- 0.00017 |
| | 2.83742 +/- 0.00072 |
| | 3.38163 +/- 0.00066 |
|  2L4P.b | -5.89794 +/- 0.00099 |
| | 0.00112 +/- 0.00095 |
|  2L6P.a | -8.64045 +/- 0.00392 |
| | -0.00220 +/- 0.00393 |
|  2L6P.b | -1.19040 +/- 0.00092 |
| | 0.00147 +/- 0.00092 |
|  2L6P.c | -7.62856 +/- 0.00716 |
| | -0.00052 +/- 0.00724 |
|  2L6P.d | -1.83639 +/- 0.00075 |
| | -0.00042 +/- 0.00075 |
|  2L6P.e | -4.61094 +/- 0.00423 |
| | 0.00404 +/- 0.00430 |
|  2L6P.f | -1.02723 +/- 0.00111 |
| | 0.00165 +/- 0.00112 |

| Topology | Numerical LTD |
|---|---|
|  1L6P | 0.51018 +/- 0.00031 |
| | -1.54768 +/- 0.00032 |
| | 0.60407 +/- 0.00216 |
| | -6.96436 +/- 0.00213 |
| | 0.40655 +/- 0.00152 |
| | -2.51588 +/- 0.00157 |
| | 1.30529 +/- 0.00289 |
| | -2.27744 +/- 0.00284 |
| | -2.20131 +/- 0.00241 |
| | -6.37841 +/- 0.00254 |
| | -1.28057 +/- 0.00088 |
| | -2.21602 +/- 0.00088 |
|  1L8P | 5.10300 +/- 0.00400 |
| | -1.62544 +/- 0.00373 |
| | 4.21309 +/- 0.00421 |
| | -1.95771 +/- 0.00394 |
| | 1.26931 +/- 0.00486 |
| | -0.84023 +/- 0.00503 |
| | -0.35626 +/- 0.00057 |
| | -1.46911 +/- 0.00058 |
| | -1.16905 +/- 0.00794 |
| | -2.72569 +/- 0.00967 |
| | -0.57605 +/- 0.00196 |
| | -4.04047 +/- 0.00202 |

| Topology | Numerical LTD |
|---|---|
|  2L4P.b | -1.08656 +/- 0.00127 |
| | 2.86702 +/- 0.00125 |
| | 3.09646 +/- 0.00696 |
| | 9.53952 +/- 0.00706 |
| | 1.70253 +/- 0.00285 |
| | 4.56488 +/- 0.00291 |
| | 2.80094 +/- 0.00023 |
| | 3.34866 +/- 0.00025 |
| | 8.15559 +/- 0.00123 |
| | 6.10277 +/- 0.00124 |
| | 3.10306 +/- 0.00021 |
| | 0.09376 +/- 0.00020 |
|  2L5P | 0.27368 +/- 0.00131 |
| | 1.44760 +/- 0.00129 |
| | 1.08568 +/- 0.00342 |
| | 1.78725 +/- 0.00339 |
| | 2.09848 +/- 0.00648 |
| | 2.04022 +/- 0.00648 |
| | 1.51586 +/- 0.00027 |
| | 1.31451 +/- 0.00027 |
| | 1.97798 +/- 0.01394 |
| | 1.13209 +/- 0.01173 |
| | 2.00638 +/- 0.00061 |
| | -0.08277 +/- 0.00060 |

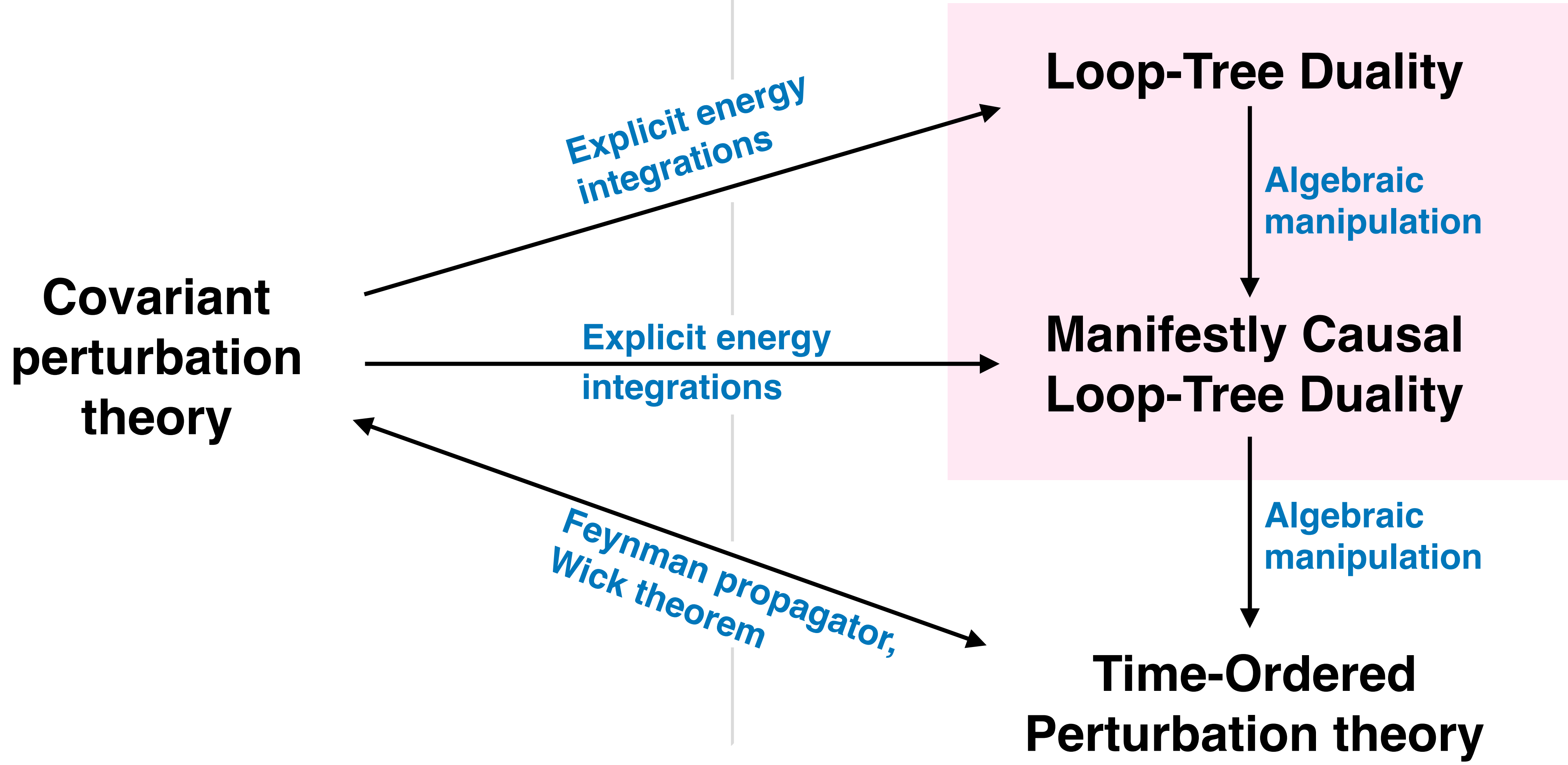
| Topology | Numerical LTD |
|---|---|
|  3L4P | 0.00796 +/- 0.00877 |
| | -6.73786 +/- 0.00856 |
|  4L4P.a | 8.38828 +/- 0.07772 |
| | -0.01028 +/- 0.07754 |
|  4L4P.b | 7.96654 +/- 0.11281 |
| | 0.07617 +/- 0.11858 |
|  5L4P | 3.28900 +/- 0.01964 |
|  6L4P.a | 8.36493 +/- 0.02167 |
|  6L4P.b | 1.09968 +/- 0.41729 |

| Topology | Numerical LTD |
|---|---|
|  3L4P | -2.43299 +/- 0.03927 |
| | -3.41797 +/- 0.03956 |
| | -5.36759 +/- 0.14110 |
| | -1.05826 +/- 0.13399 |
| | -4.46226 +/- 0.10022 |
| | -0.72941 +/- 0.09918 |
| | -3.89588 +/- 0.00173 |
| | 3.89127 +/- 0.00165 |
| | -3.15581 +/- 0.00639 |
| | 2.97368 +/- 0.00633 |
| | -0.10876 +/- 0.00096 |
| | 1.86939 +/- 0.00095 |
|  3L5P | -1.06298 +/- 0.02843 |
| | -0.88557 +/- 0.02875 |
| | -3.28794 +/- 0.07308 |
| | -0.29022 +/- 0.07635 |
| | -1.61475 +/- 0.14277 |
| | 0.25654 +/- 0.13621 |
| | -1.26220 +/- 0.00124 |
| | 1.06124 +/- 0.00123 |
| | 4.58640 +/- 0.00609 |
| | 1.80523 +/- 0.00645 |
| | -1.05359 +/- 0.01706 |
| | 5.92117 +/- 0.01660 |
|  4L4P.a | 1.28725 +/- 0.00637 |
| | 2.95568 +/- 0.00642 |
|  4L4P.b | -4.34119 +/- 0.01166 |
| | -2.77244 +/- 0.01160 |

# Other representations…

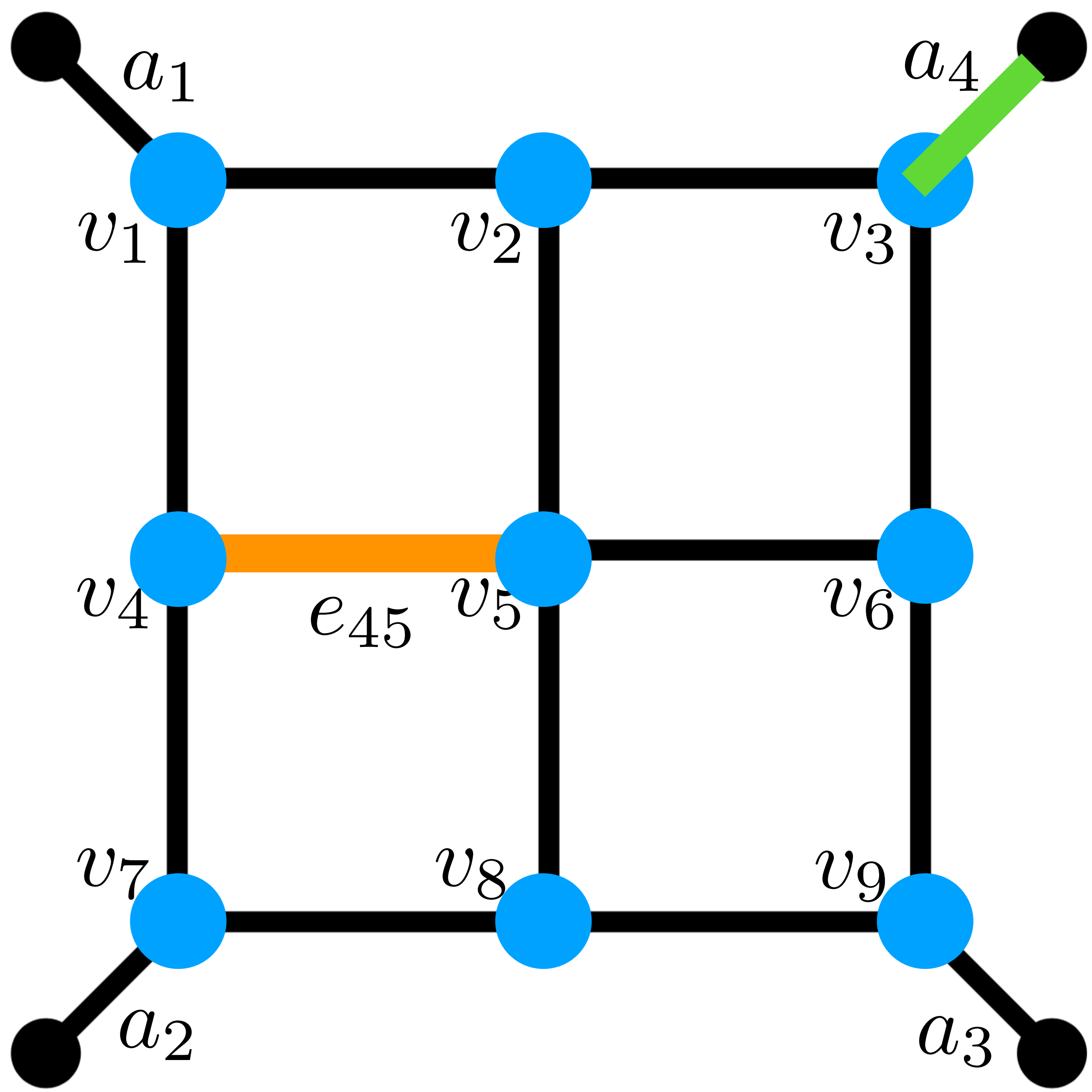
$$I = \int \left( \prod_{i=1}^{L} \frac{d^4 k_i}{(2\pi)^4} \right) \frac{N}{\prod_{i\in \mathbf{e}} (q_i^2 - m_i^2)}$$

$$I = \int \left( \prod_{i=1}^{L} \frac{d^3 \vec{k}_i}{(2\pi)^3} \right) f$$




The Loop Tree duality offers the best understanding of IR singularities and their cancellations, other than being relatively efficient to evaluate

# Just a bit of notation…




We give to each internal vertex a label

$$v_i, \ \ i = 1, ..., 9$$

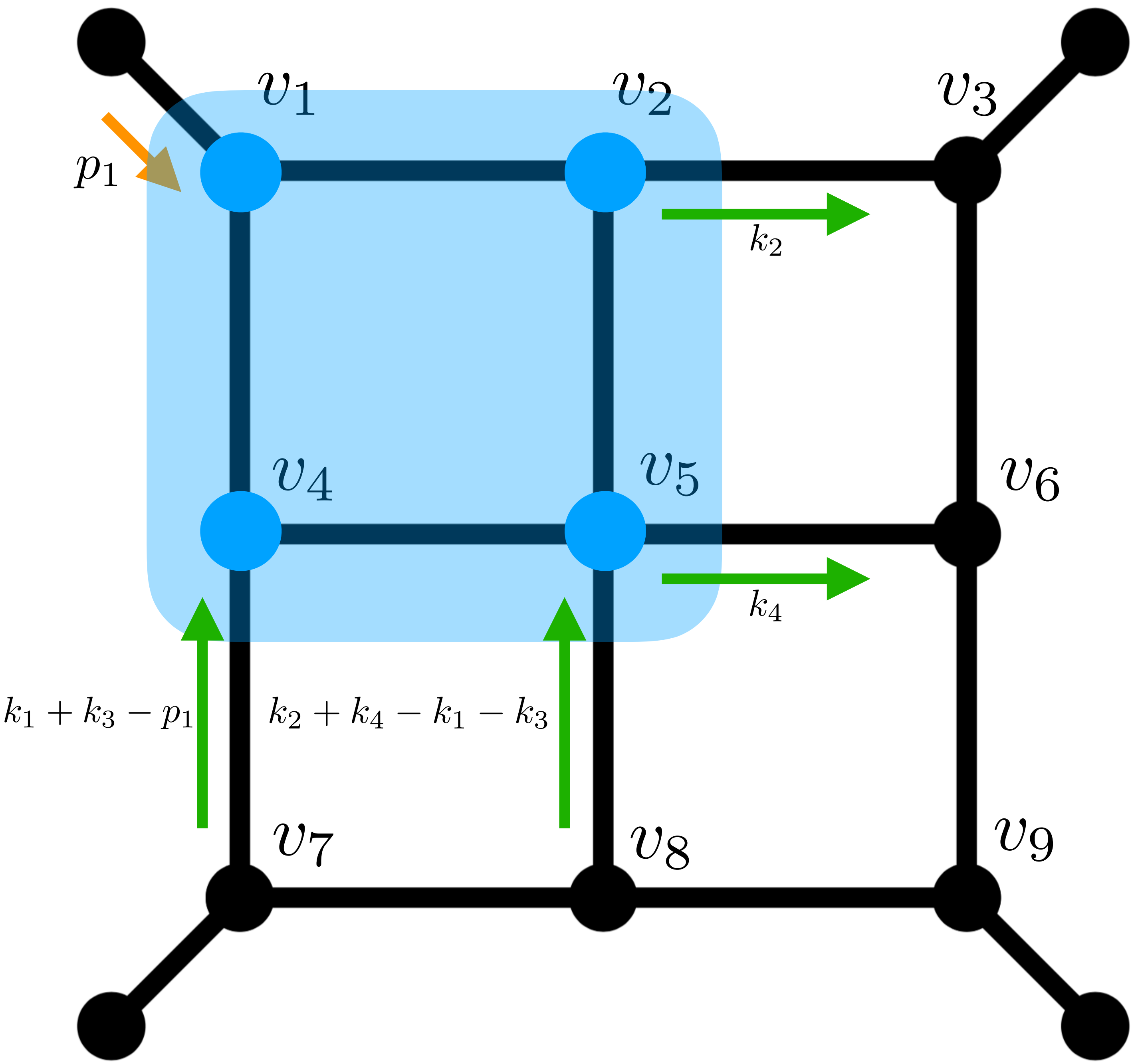
Each internal edge corresponds to a couplet of vertices

$$e_{ij} = \{v_i, v_j\}$$

External edges are denoted as

$$a_i, \ \ i = 1, ..., 4$$

# Physical Thresholds as Connected Cuts



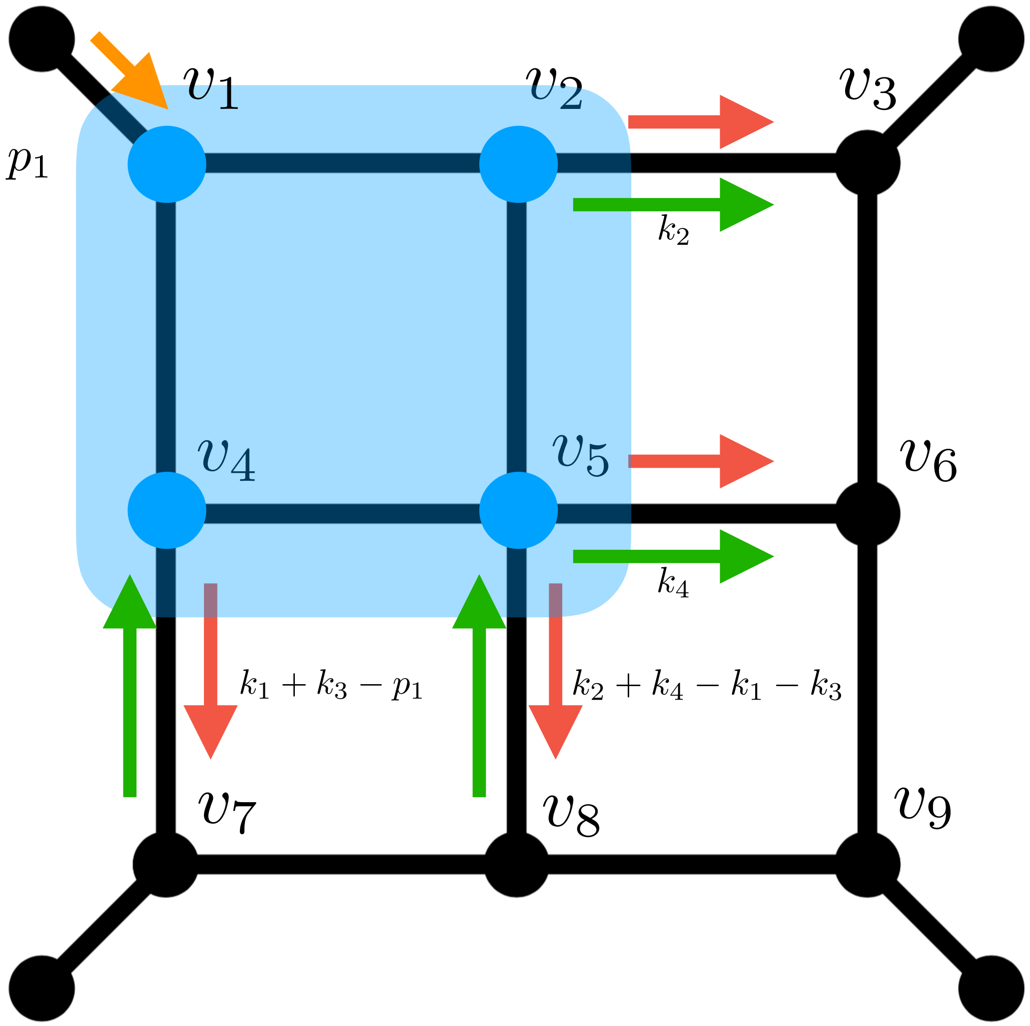

Green arrow: momentum orientation

$$\mathbf{s} = \{v_1, v_2, v_4, v_5\}$$

The boundary of this set contains all the edges connecting vertices in it with vertices outside of it

$$\delta(\mathbf{s}) = \{e_{23}, e_{56}, e_{58}, e_{47}, a_1\}$$

**This set completely characterises a threshold**




Draw **energy-flow arrows** by **flipping** the green arrows that flow inside the set

Denote by

$$E_{e_{12}} = \sqrt{|\vec{k}_1|^2 + m_{12}^2}$$

the on-shell energy of $e_{12}$

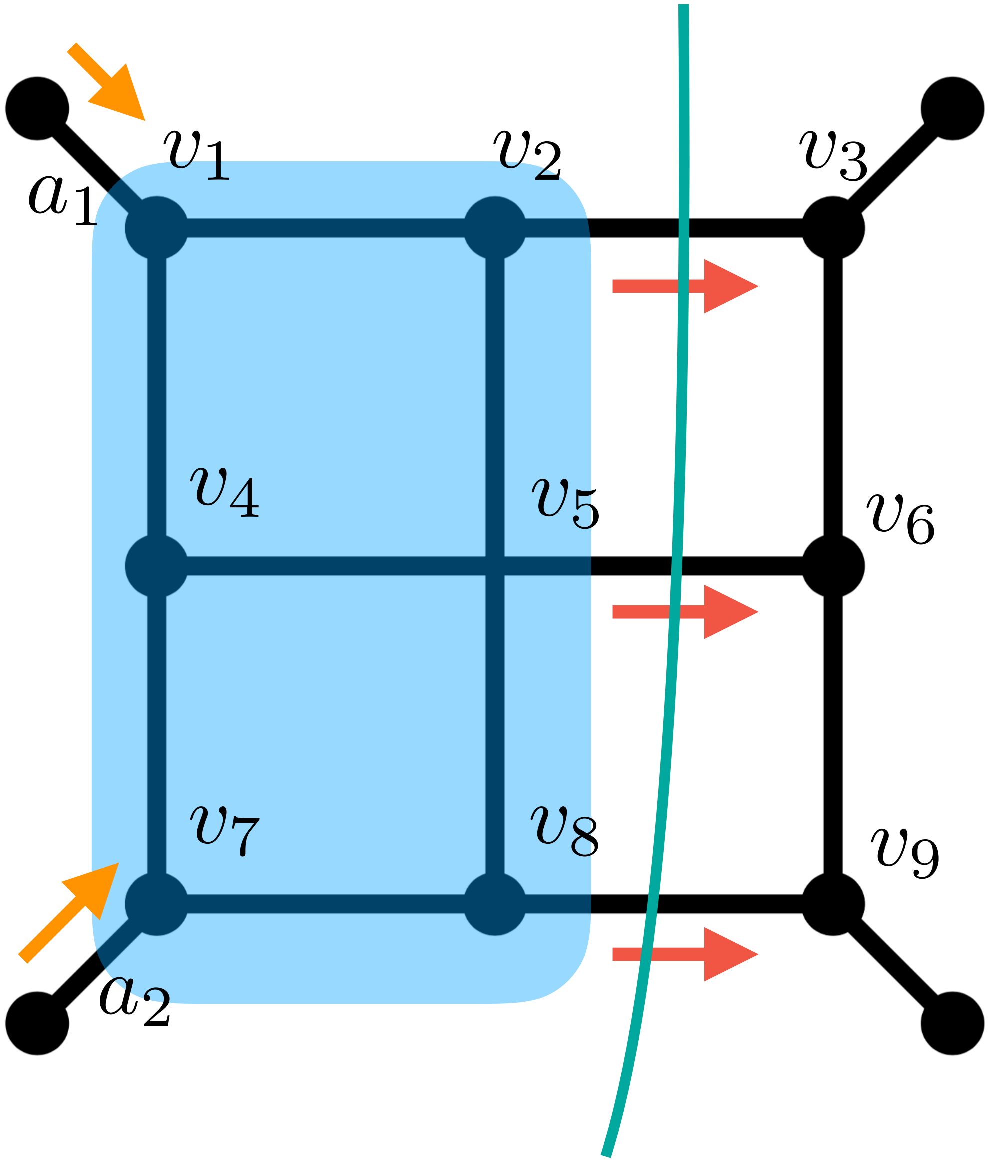
Reading the **conservation of on-shell energies** for particles going in/out of the set

$$\eta_{\mathbf{s}} = E_{e_{23}} + E_{e_{56}} + E_{e_{58}} + E_{e_{47}} - E_{a_1} = 0$$

or, if we want to be fancy…

$$\eta_{\mathbf{s}} = \sum_{e \in \delta(\mathbf{s}) \setminus \mathbf{e}_{\text{ext}}} E_e - \sum_{e \in \mathbf{a}} E_e + \sum_{e \in \mathbf{e}_{\text{ext}} \setminus \mathbf{a}} E_e$$

# S-channel thresholds and Cutkosky cuts




Consider a **specific subclass of connected cuts**, those whose boundary contains $a_1, \ a_2$

Let $\mathbf{s} \subset \mathbf{v}$ such that

- $\mathbf{s}, \ \mathbf{v} \setminus \mathbf{s}$ are **connected**
- $\delta(\mathbf{s}) \cap \mathbf{e}_{\text{ext}} = \{a_1, a_2\}$

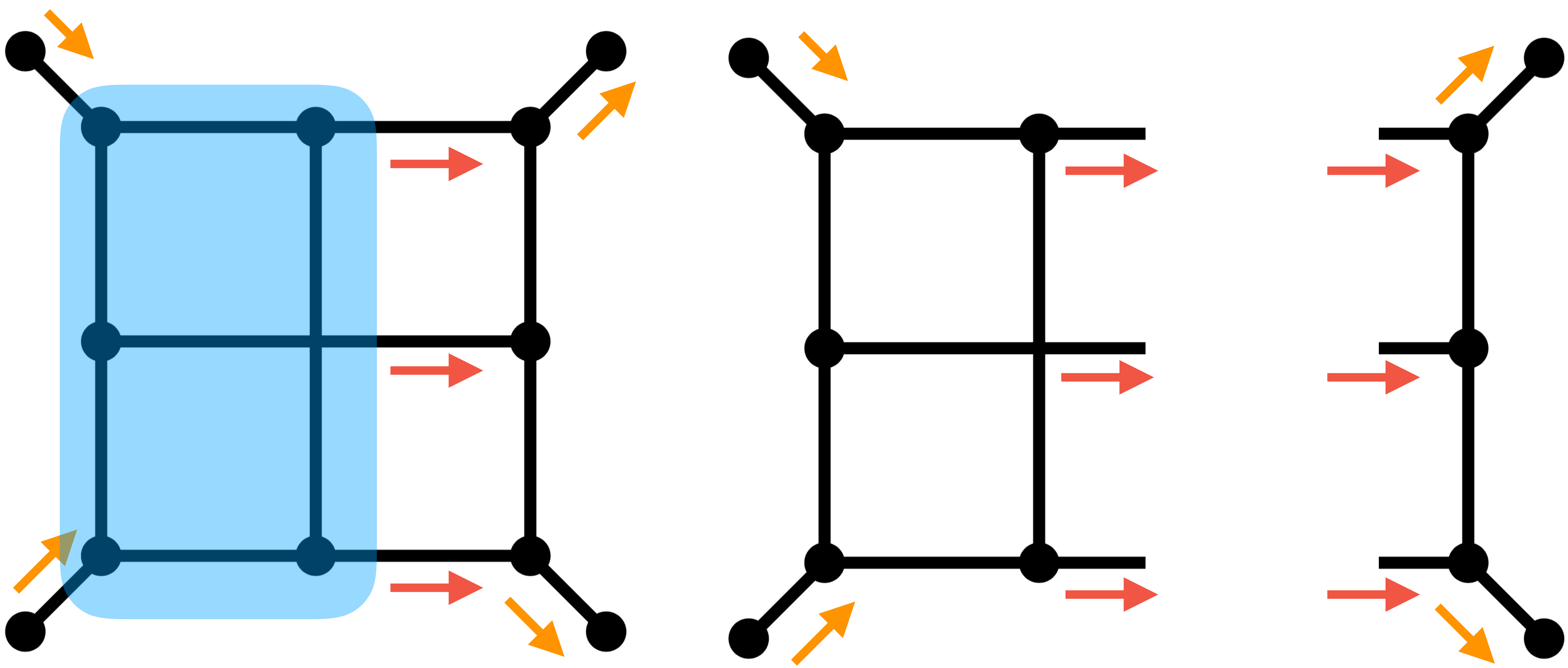
An example:

$$\mathbf{s} = \{v_1, v_2, v_4, v_5, v_7, v_8\}$$

The **Cutkosky cut** can be denoted by a line crossing the internal edges in $\delta(\mathbf{s})$

$$\mathbf{c}_\mathbf{s} = \delta(\mathbf{s}) \setminus \mathbf{e}_{\text{ext}} = \{e_{23}, e_{56}, e_{89}\}$$

**We have just constructed an interference diagram from a "bigger" graph rather than as a product of amplitudes**

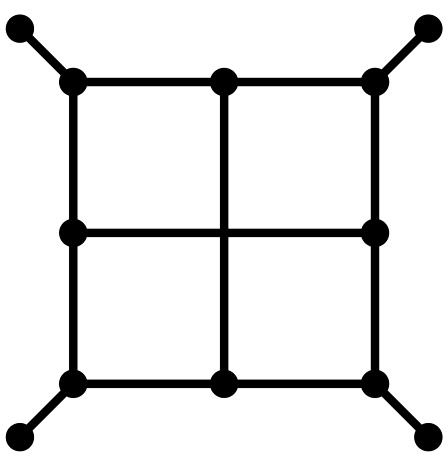


# LSZ, Cutkosky cuts, and how to construct an interference diagram

In a way, we already knew this description of thresholds…

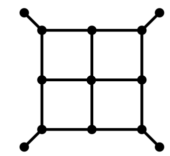
- Interference diagrams are obtained by **contour deforming certain thresholds**.
- In **LSZ**, interference diagrams are obtained by **glueing connected amplitudes**.

**In order to formulate and connect these two principles rigorously, we need the LTD representation!**

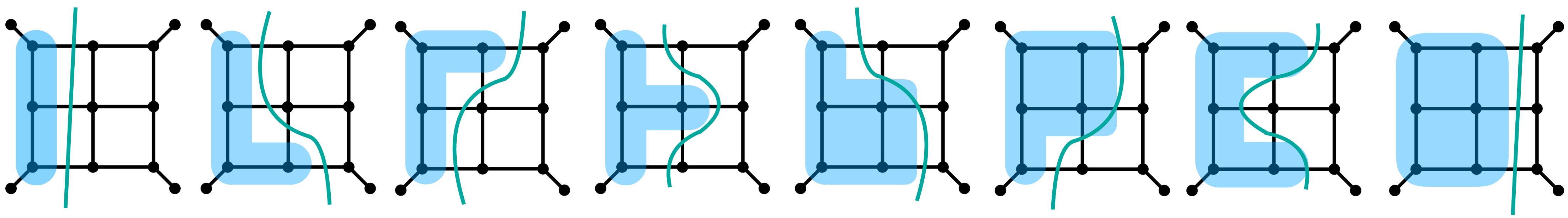


- Consider this graph, called the **supergraph**
- **Construct interference diagrams from it**, by summing over the thresholds of its LTD representation

# A (rough) recipe to construct cross sections

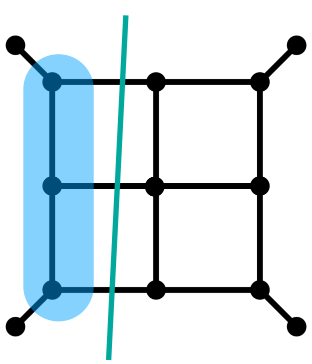
Consider the LTD representation of 

1. Its thresholds correspond to connected cuts




2. Associate a Cutkosky cut to any s-channel threshold

**Cutkosky cut**

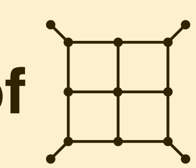


$$= \int \left( \prod_{i=1}^{L} \frac{d^3 \vec{k}}{(2\pi)^3} \right) f_{\text{ltd}}(\boxplus) \; \eta_{\mathbf{s}_1} \delta(\eta_{\mathbf{s}_1}) \; \mathcal{O}_{\mathbf{s}_1}$$

**Observable**

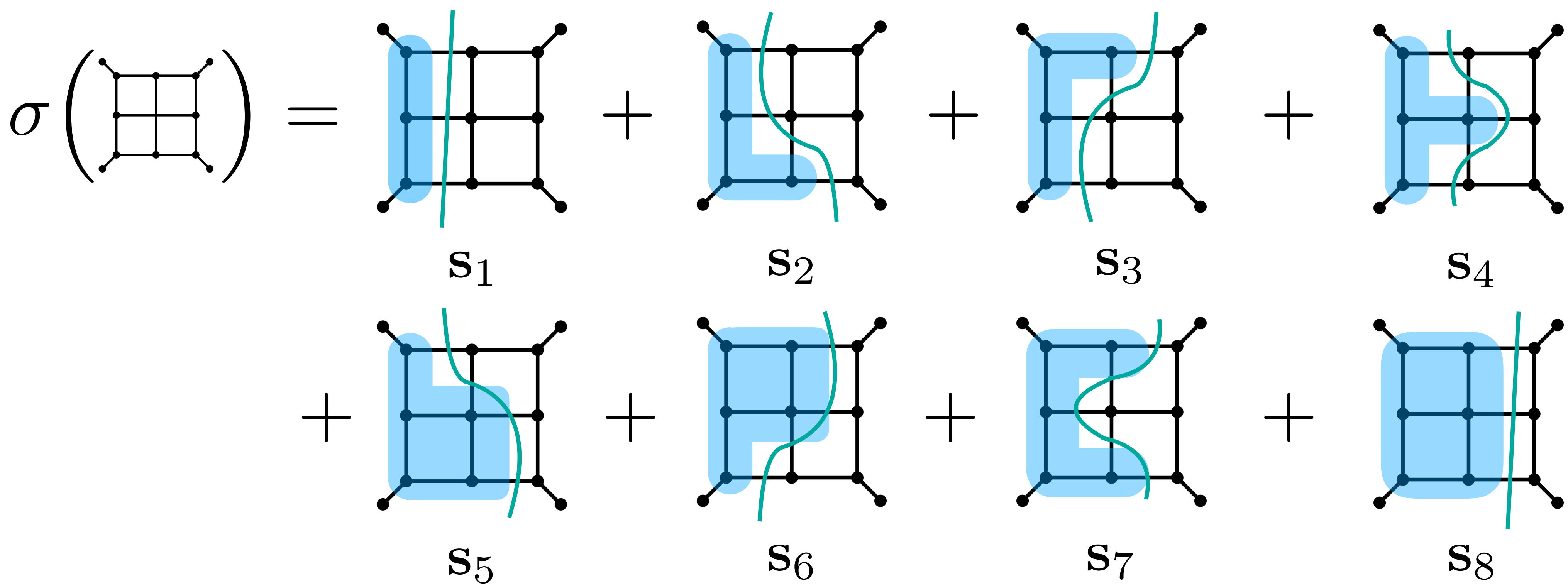
$\mathbf{s}_1$

*(but there are some subtleties as we will see)*

**LTD representation of** 

3. Keep a **consistent loop momentum** routing between all interference diagrams

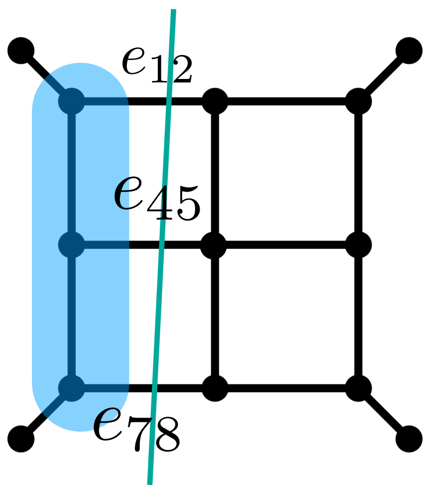
## Sum all these interference diagrams together to obtain the cross-section per super graph



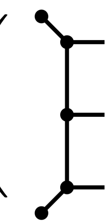

## We will now show that this sum is free of IR singularities

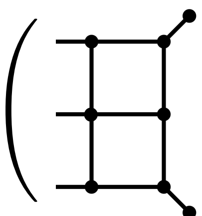
# Cancellation of thresholds

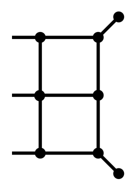
The formula shown before can be manipulated to obtain



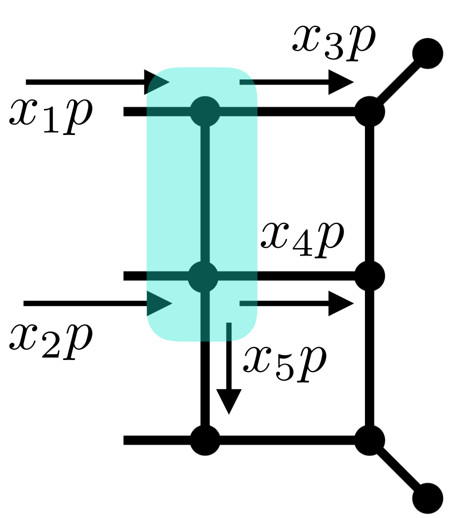

$$= \int \left( \prod_{i=1}^{L} \frac{d^3\vec{k}}{(2\pi)^3} \right) \frac{\delta(E_{e_{12}} + E_{e_{45}} + E_{e_{78}} - Q_0)}{2E_{e_{12}} 2E_{e_{45}} 2E_{e_{78}}} f_{\rm ltd}\left( \cdot \right) f_{\rm ltd}\left( \cdot \right)$$





The singularities of $f_{\rm ltd}\left( \cdot \right)$  are themselves connected sets!

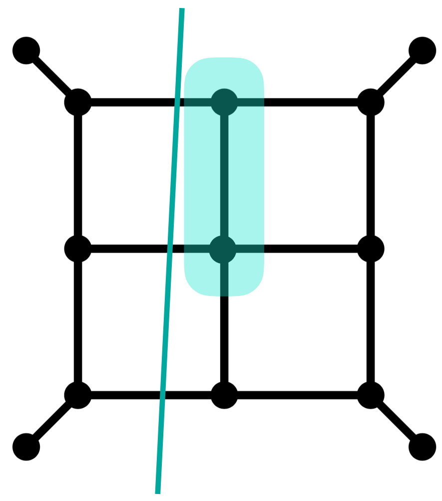
e.g.

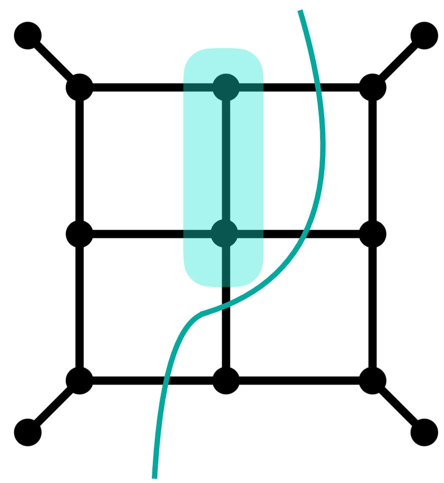



The cluster of collinear particles going in and outside the set are **degenerate** at the singular points

# The cancelling partner

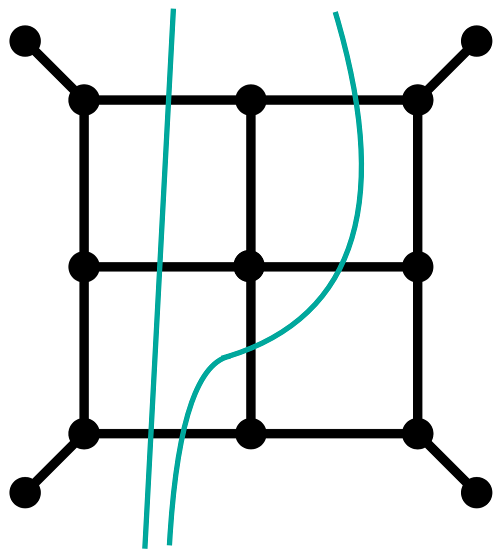
How do we find the contribution cancelling this singularity? It's all about **degeneracy**





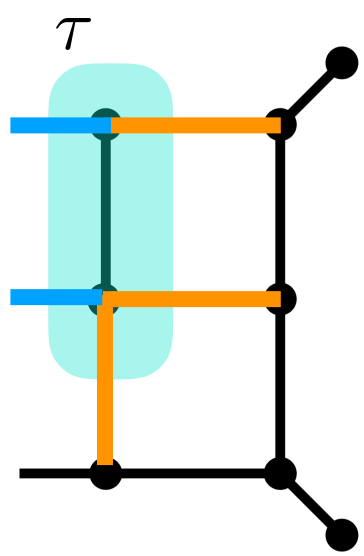
Just move the Cutkosky cut across the singularity!

**At the location of the singularity, these two interference diagrams become the same**



# Showing cancellations

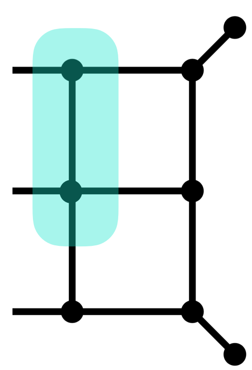
Consider the internal particles in the boundary of the cut

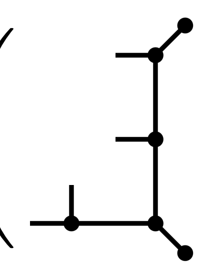
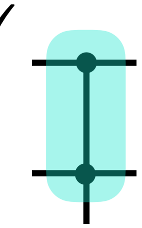



$$\delta(\tau) = \{e_{23}, e_{56}, e_{58}, e_{12}, e_{45}\}$$

**Particles internal to the amplitude** ($e_{23}, e_{56}, e_{58}$) — **Particles external to the amplitude** ($e_{12}, e_{45}$)

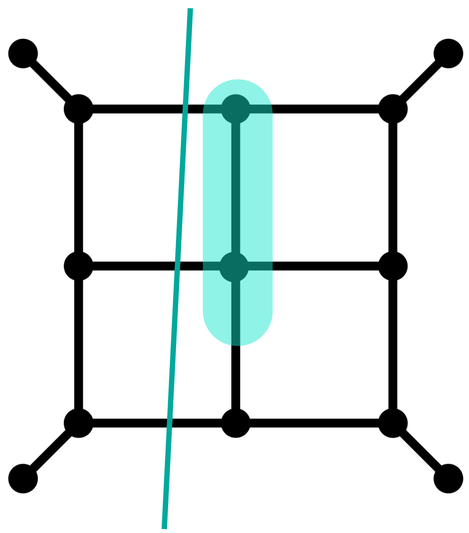
The LTD representation factorises in the product of the LTD representation of the two smaller amplitudes and the threshold

$f_{\rm ltd}\Big($  $\Big) \approx$

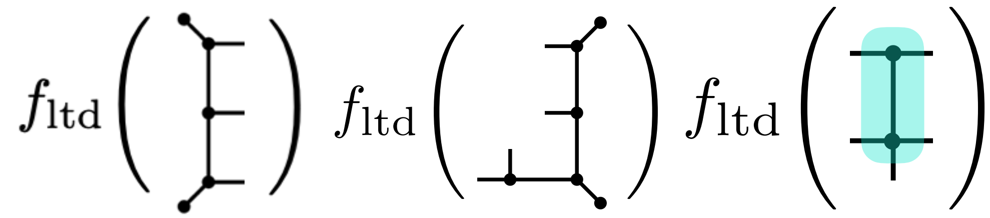
$f_{\rm ltd}\Big($  $\Big)\, f_{\rm ltd}\Big($  $\Big)$

divided by

$$2E_{e_{23}}2E_{e_{56}}2E_{e_{58}}(E_{e_{23}}+E_{e_{56}}+E_{e_{58}}-E_{e_{12}}-E_{e_{45}})$$

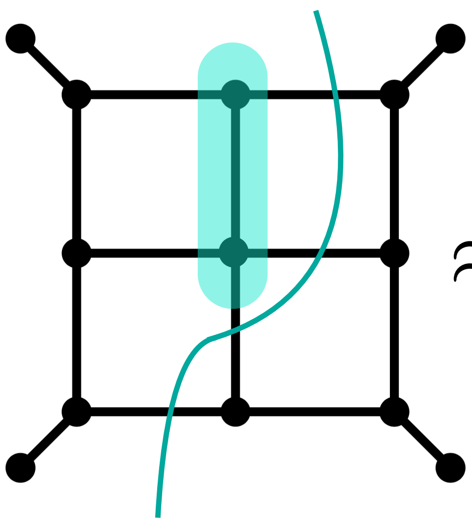


$$\approx \frac{\delta(E_{e_{12}}+E_{e_{45}}+E_{e_{78}}-Q_0)}{2E_{e_{12}}2E_{e_{45}}2E_{e_{78}}}$$

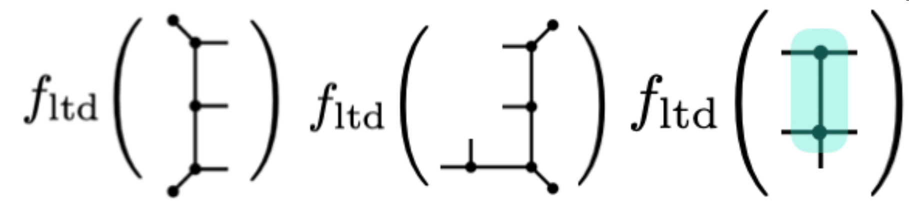



$$2E_{e_{23}}2E_{e_{56}}2E_{e_{58}}(E_{e_{23}}+E_{e_{56}}+E_{e_{58}}-E_{e_{12}}-E_{e_{45}})$$

**Same singularity, opposite sign!**



$$\approx \frac{\delta(E_{e_{23}}+E_{e_{56}}+E_{e_{58}}+E_{e_{78}}-Q_0)}{2E_{e_{23}}2E_{e_{56}}2E_{e_{58}}2E_{e_{78}}}$$




$$2E_{e_{45}}2E_{e_{12}}(E_{e_{12}}+E_{e_{45}}-E_{e_{23}}-E_{e_{56}}-E_{e_{58}})$$

Everything a part from the delta is manifestly the same. If we substitute

$$E_{e_{12}}+E_{e_{45}}-E_{e_{23}}-E_{e_{56}}-E_{e_{58}}=0$$

$$\delta(E_{e_{12}}+E_{e_{45}}+E_{e_{78}}-Q_0)\to\delta(E_{e_{23}}+E_{e_{56}}+E_{e_{58}}+E_{e_{78}}-Q_0)$$

# Causal flow or constructing a local representation

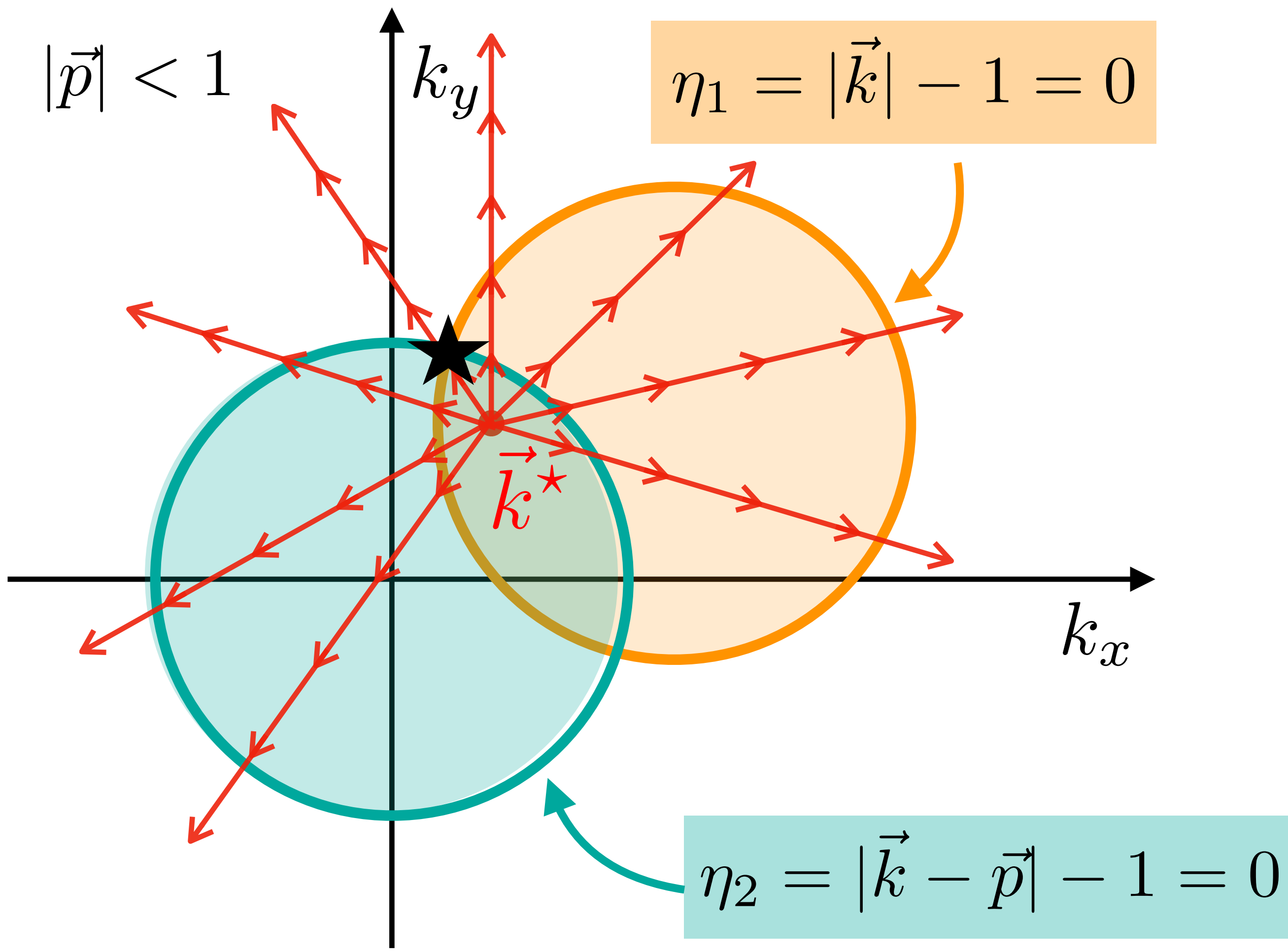
What we said up until now does not address how to construct the actual local representation, which requires **solving the deltas**!

$$\int d\vec{k}\; \frac{\delta(|\vec{k}|-1)}{|\vec{k}-\vec{p}|-1} + \frac{\delta(|\vec{k}-\vec{p}|-1)}{|\vec{k}|-1}$$

$$1 = \int dt h(t)$$

$$= \int dt \int d\vec{k}\; \frac{h(t)\delta(|\vec{k}|-1)}{|\vec{k}-\vec{p}|-1} + \frac{h(t)\delta(|\vec{k}-\vec{p}|-1)}{|\vec{k}|-1}$$

$$= \int dt \int d\vec{k}\mathbb{J}\phi \left( \frac{h(t)\delta(|\phi(t,\vec{k})|-1)}{|\phi(t,\vec{k})-\vec{p}|-1} + \frac{h(t)\delta(|\phi(t,\vec{k})-\vec{p}|-1)}{|\phi(t,\vec{k})|-1} \right)$$




Choose:

$$\begin{cases} \partial_t \phi(t, \vec{k}) = \kappa(\phi(t, \vec{k})) \\ \phi(0, \vec{k}) = \vec{k} \end{cases}$$

With:

$$\kappa \cdot \nabla \eta_i > 0 \text{ if } \eta_i = 0$$

$\kappa$ is the field used to contour deform around thresholds!

$$\kappa = \vec{k} - \vec{k}^\star$$

Then $\forall \vec{k}$ $\exists!\ t_i^\star \in \mathbb{R}$ s.t.

$$|\phi(t_1^\star, \vec{k})| - 1 = 0$$

$$|\phi(t_2^\star, \vec{k}) - \vec{p}| - 1 = 0$$

Points on different thresholds are **correlated**, so that **cancelling partners are evaluated at the same point when they need to cancel**!

$$= \int dt \int d\vec{k} \mathbb{J}\phi \left( \frac{h(t)\delta(|\phi(t,\vec{k})| - 1)}{|\phi(t,\vec{k}) - \vec{p}| - 1} + \frac{h(t)\delta(|\phi(t,\vec{k}) - \vec{p}| - 1)}{|\phi(t,\vec{k})| - 1} \right)$$

$$= \int dt \int d\vec{k} \mathbb{J}\phi \left( \frac{h(t_1^\star)}{\partial_t |\phi(t_1^\star,\vec{k})| (|\phi(t_1^\star,\vec{k}) - \vec{p}| - 1)} + \frac{h(t_2^\star)}{\partial_t |\phi(t_2^\star,\vec{k}) - \vec{p}| (|\phi(t_2^\star,\vec{k})| - 1)} \right)$$

Look at the singularities of the first term. It is exactly the equation defining $t_2^\star$

$$|\phi(t_1^\star,\vec{k}) - \vec{p}| - 1 = 0 \quad \Rightarrow \quad t_1^\star = t_2^\star \qquad \begin{pmatrix} |\phi(t_1^\star,\vec{k})| - 1 = 0 \\ |\phi(t_2^\star,\vec{k}) - \vec{p}| - 1 = 0 \end{pmatrix}$$

Furthermore

$$|\phi(t_1^\star,\vec{k}) - \vec{p}| - 1 = |\phi(t_2^\star,\vec{k}) - \vec{p}| - 1 + (t_1^\star - t_2^\star)\partial_t |\phi(t_2^\star,\vec{k}) - \vec{p}| + o((t_1^\star - t_2^\star))$$

$$= (t_1^\star - t_2^\star)\partial_t |\phi(t_2^\star,\vec{k}) - \vec{p}| + o((t_1^\star - t_2^\star))$$

which is a **simple pole in the flow variable**! Expanding carefully…

Expanding carefully the two term composing the integrand…

$$\frac{h(t_1^\star)}{\partial_t|\phi(t_1^\star,\vec{k})|(|\phi(t_1^\star,\vec{k})-\vec{p}|-1)} = \frac{h(t_2^\star)}{\partial_t|\phi(t_1^\star,\vec{k})|\partial_t|\phi(t_2^\star,\vec{k})-\vec{p}|(t_1^\star-t_2^\star)} + \mathcal{O}((t_1^\star-t_2^\star)^0)$$

$$\frac{h(t_2^\star)}{\partial_t|\phi(t_2^\star,\vec{k})-\vec{p}|(|\phi(t_2^\star,\vec{k})|-1)} = \frac{h(t_2^\star)}{\partial_t|\phi(t_2^\star,\vec{k})-\vec{p}|\partial_t|\phi(t_1^\star,\vec{k})|(t_2^\star-t_1^\star)} + \mathcal{O}((t_1^\star-t_2^\star)^0)$$

So that **they combine to a finite quantity**!!!

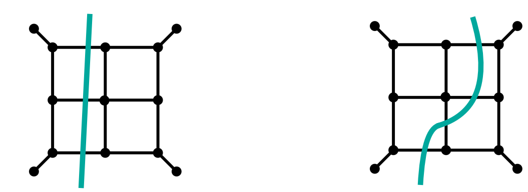
$$\frac{h(t_2^\star)}{\partial_t|\phi(t_2^\star,\vec{k})-\vec{p}|(|\phi(t_2^\star,\vec{k})|-1)} + \frac{h(t_1^\star)}{\partial_t|\phi(t_1^\star,\vec{k})|(|\phi(t_1^\star,\vec{k})-\vec{p}|-1)} = \mathcal{O}((t_1^\star-t_2^\star)^0)$$

# IR safety

- $t$ is a **natural parameter in which to expand** to show cancellations
- One single parameter to approach all limits (single/double collinear, soft collinear etc. )
- Parameter in which we solve the deltas $\Rightarrow$ **1d residue theorem along the flow!**

This same expansion can be performed for the interference diagrams

**The major difference is the observable!**

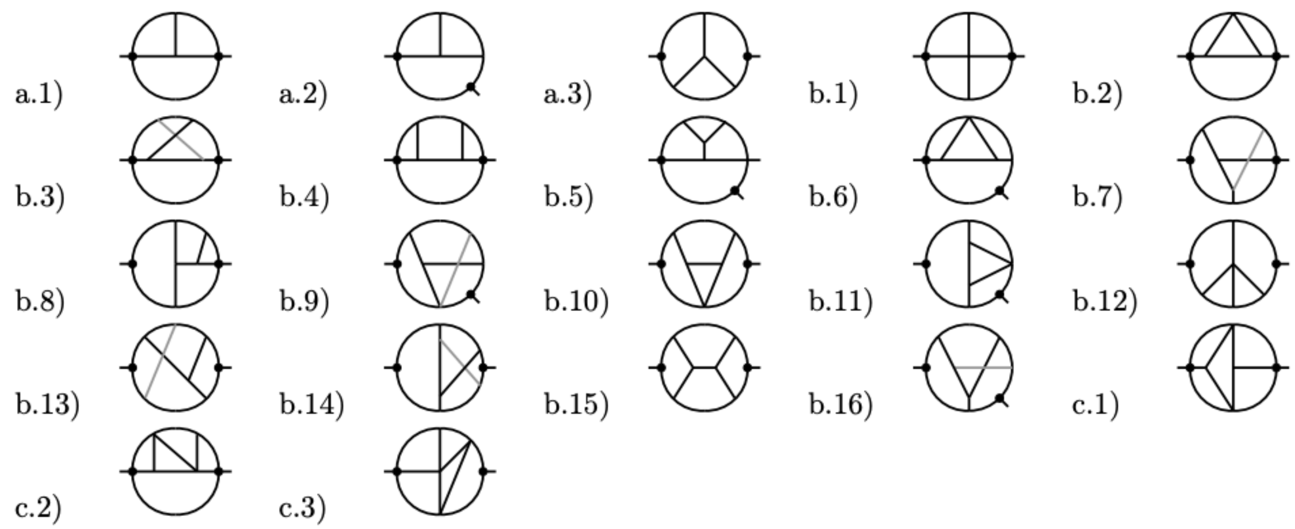


Finite-sized neighbourhood

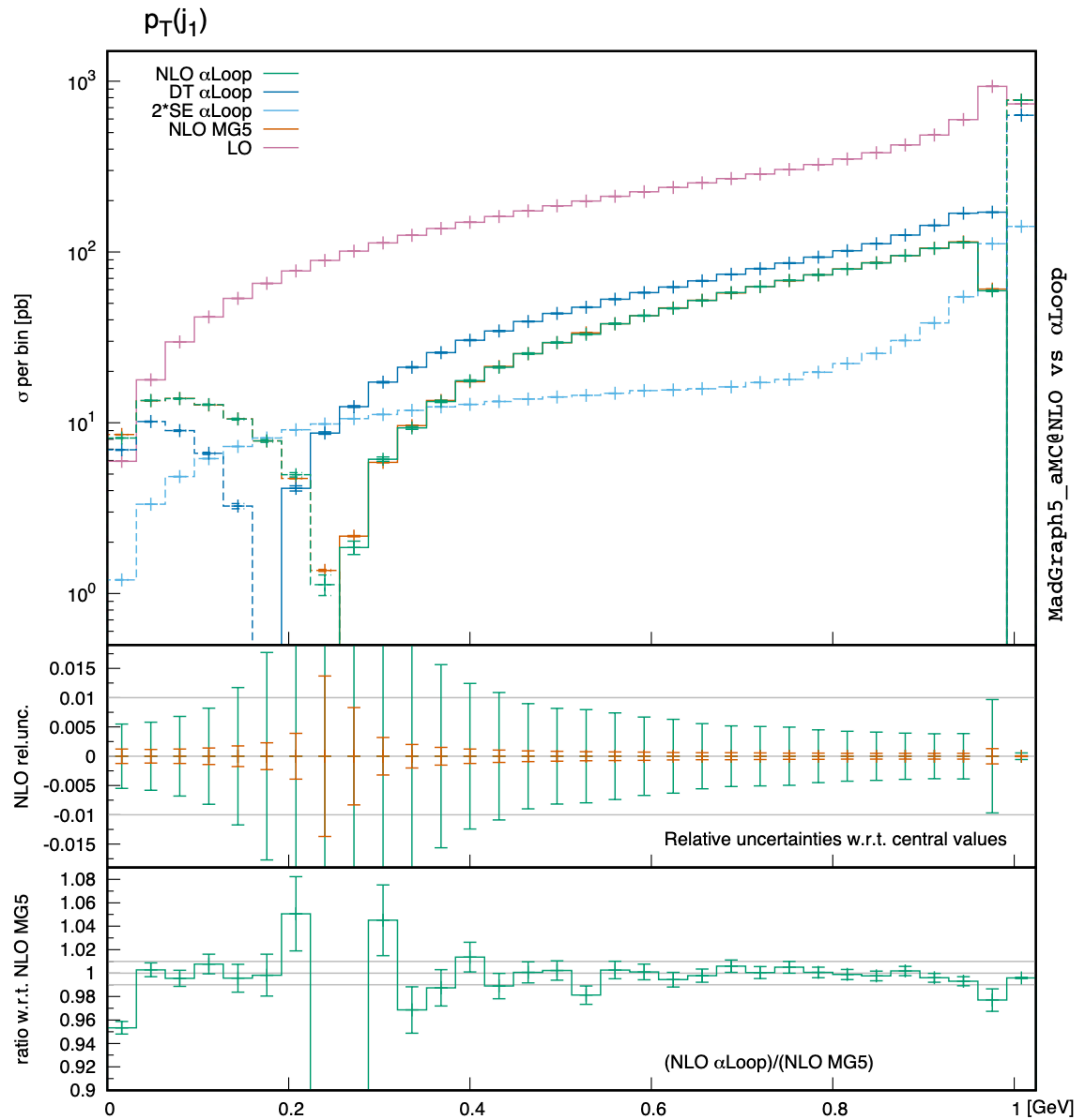
**Requiring IR safety** $\Longrightarrow$ $$\mathcal{O}_{\mathbf{s}_1}(\vec{k}) = \mathcal{O}_{\mathbf{s}_6}(\vec{k}), \ \forall \vec{k} \in B_\epsilon(S)$$

Soft/collinear region

$\epsilon$ is a **mathematically needed scale**, gauging the volume of phase space in which the observables must coincide $\Longrightarrow$ **Experimental resolution** of degenerate parton configurations!

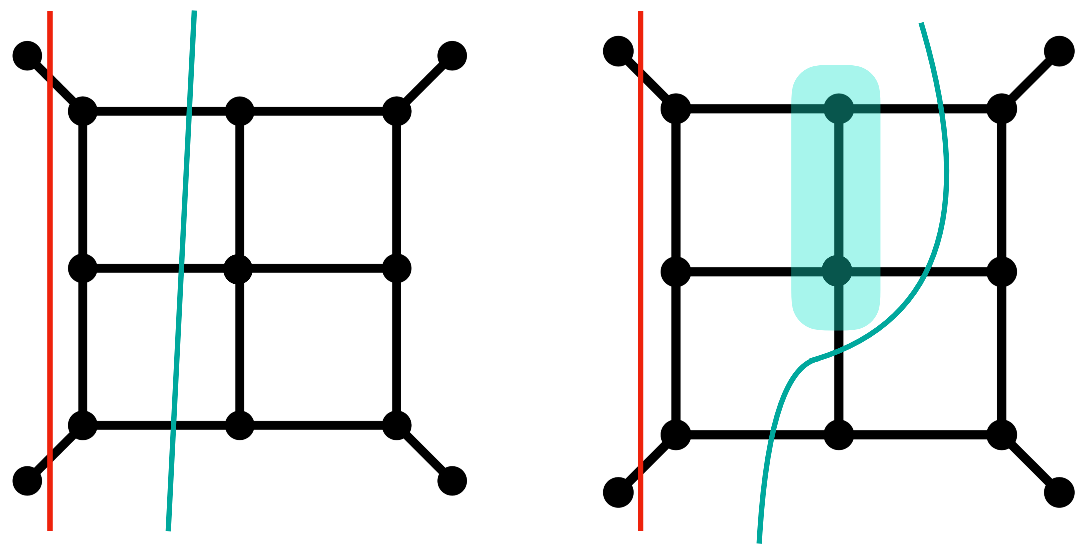



| Γ | $N_p$ [$10^6$] | t/p [μs] min | t/p [μs] avg | $N_{ch}$ | FORCER [$GeV^2$] | αLOOP [$GeV^2$] | exp. | Δ [σ] | Δ [%] |
|---|---|---|---|---|---|---|---|---|---|
| Inclusive cross-section per supergraph | | | | | | | | | |
| a.1 | 1 | 5 | 450 | 16 | 5.75396 | 5.7530(46) | -6 | 0.21 | 0.00017 |
| a.2 | 1 | 10 | 690 | 16 | -5.75396 | -5.763(11) | -6 | 0.82 | 0.0016 |
| a.3 | 1 | 25 | 1400 | 16 | -5.75396 | -5.771(23) | -6 | 0.74 | 0.0039 |
| b.1 | 1 | 150 | 6600 | 45 | -1.04773 | -1.0459(23) | -7 | 0.79 | 0.0017 |
| b.2 | 1 | 270 | 39000 | 45 | -1.04773 | -1.0457(21) | -7 | 0.97 | 0.0029 |
| b.3 | 1 | 320 | 52000 | 81 | -1.04773 | -1.0448(21) | -7 | 1.4 | 0.0028 |
| b.4 | 1 | 740 | 96000 | 75 | -1.04773 | -1.0455(22) | -7 | 1.0 | 0.0021 |
| b.5 | 1 | 340 | 20000 | 45 | -1.04773 | -1.0441(23) | -7 | 1.6 | 0.0035 |
| b.6 | 1 | 350 | 12000 | 45 | -1.04773 | -1.0434(26) | -7 | 1.7 | 0.0042 |
| b.7 | 1 | 1800 | 180000 | 81 | -1.04773 | -1.0563(51) | -7 | 1.7 | 0.0081 |
| b.8 | 1 | 1400 | 120000 | 75 | -1.04773 | -1.0526(42) | -7 | 1.2 | 0.0046 |
| b.9 | 1 | 1200 | 36000 | 45 | -1.04773 | -1.0439(27) | -7 | 1.4 | 0.0037 |
| b.10 | 1 | 1100 | 32000 | 45 | -1.04773 | -1.0488(29) | -7 | 0.37 | 0.0010 |
| b.11 | 1 | 1100 | 54000 | 45 | -1.04773 | -1.0516(35) | -7 | 1.1 | 0.0037 |
| b.12 | 1 | 1100 | 30000 | 45 | -1.04773 | -1.0473(30) | -7 | 0.14 | 0.00041 |
| b.13 | 1 | 2700 | 83000 | 45 | -1.04773 | -1.040(15) | -7 | 0.51 | 0.0074 |
| b.14 | 1 | 3100 | 110000 | 75 | -2.09546 | -2.123(12) | -7 | 2.3 | 0.0130 |
| b.15 | 1 | 3100 | 210000 | 81 | -2.09546 | -2.1045(67) | -7 | 1.3 | 0.0043 |
| b.16 | 2 | 1800 | 120000 | 75 | -5.23865 | -5.312(65) | -8 | 1.1 | 0.014 |
| c.1 | 1 | 1100 | 49000 | 128 | 1.66419 | 1.6691(79) | -9 | 0.62 | 0.0029 |
| c.2 | 1 | 900 | 46000 | 130 | 1.77832 | 1.7752(71) | -9 | 0.44 | 0.0018 |
| c.3 | 1 | 1600 | 69000 | 130 | 1.77832 | 1.7797(33) | -9 | 0.42 | 0.00077 |



$p_T(j_1)$
NLO αLoop
DT αLoop
2*SE αLoop
NLO MG5
LO
σ per bin [pb]
MadGraph5_aMC@NLO vs αLoop
NLO rel.unc.
Relative uncertainties w.r.t. central values
ratio w.r.t. NLO MG5
(NLO αLoop)/(NLO MG5)
0 0.2 0.4 0.6 0.8 1 [GeV]

# Initial State Radiation



Interference diagrams that cancel at the location of a singularity correspond to **varying final state multiplicities**

- If we want **ISR cancellations**, we need to consider diagrams with **more than two initial states or it is not IR-safe**!

- Furthermore, to cancel **singularities that correlate initial and final states**, we also need diagrams with disconnected amplitudes, **contradicting LSZ**

**The interference diagrams are now obtained by cutting vacuum graphs!**

How do PDF renormalisation and resummation fit into this model?
We'll have to wait to know for sure…

# A recap:

- Loop-Tree Duality representation for the sunrise
- Singularities of the sunrise
- Physical thresholds as connected cuts
- Constructing interference diagrams from the super-graph
- Easy cancellations of IR singularities through local factorisation of amplitudes
- The causal flow and hints at a general proof
- IR-safety and observables
- Initial state radiation

# References:


**Loop-Tree Duality**

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- I. Bierenbaum, S. Catani, P. Draggiotis, G. Rodrigo, A Tree-Loop Duality Relation at Two Loops and Beyond, JHEP 10 (2010) 073
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- ZC, V. Hirschi, D. Kermanschah, A. Pelloni, B. Ruijl, Numerical Loop-Tree Duality: contour deformation and subtraction, JHEP 04 (2020) 096

**Local Unitarity**

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