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Spin, the Little Group and Spinor-Helicity

INFORMAL SCIENCE COFFEE 12 JANUARY 2021- ANDREW LIFSON

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Aim and Reason for these Seminars

To try and share the interesting knowledge we have gained during my PhD

First (this) seminar:

- Very informal (questions and interruptions welcome!)
- Aim to discuss fundamental physics that I only recently understood
- Hopefully useful to all
- Main topics are:
  - Wigner's Quantum Mechanics incl. little group
  - Spinor-helicity formalism
  - How they link together to calculate amplitudes

Second (next) seminar:

- More of a standard seminar
- Show our unique research (chirality-flow method)
  - Go from Feynman diagram to number via flow lines
  - Requires minimal work compared to standard methods
- Builds on many concepts from first seminar
- Is really cool

Wigner's Quantum Mechanics

Wigner's Quantum Mechanics

How to classify different types of particles?

- Consider particle (state vector) ψ with Poincaré the only symmetry
  - Lorentz transformations: $p^\mu \to \Lambda^\mu{}_\nu p^\nu$, $U(\Lambda) = e^{-\frac{i}{2}J^{\mu\nu}\omega_{\mu\nu}}$
  - Translations: $p^\mu \to p^\mu + a^\mu$, $U(a) = e^{-iP^\mu a_\mu}$
  - Generic representation of Poincaré transformation $U(\Lambda, a)$
- What information can we know about this particle???
  - QN from translation p
  - QN from Lorentz transformations σ
- Diagnonalise translation operator $P^\mu \psi_{p,\sigma} = p^\mu \psi_{p,\sigma}$

Particle Definition in Wigner's Quantum Mechanics

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- How to classify different types of particles $\psi_{p,\sigma}$?
- Define 'standard', simple momentum k^μ (e.g. $k^\mu = (m, 0, 0, 0)$)
- Define Lorentz transformation $L(p, k)$ s.t. $p^\mu = L(p, k)^\mu{}_\nu k^\nu$

Definition of a particle

$$\psi_{p,\sigma} = N(p)U(L(p, k))\psi_{k,\sigma}$$

- $N(p) \equiv$ normalisation (not important in this discussion, will be ignored)
- $U(L(p, k))$ s.t. ψ can have any representation of Lorentz group
- σ quantum number(s) unchanged by definition

Wigner Rotations and the Little Group

How to classify different types of particles $\psi_{p,\sigma}$?

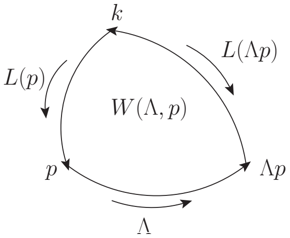
- Consider Lorentz transformation $U(\Lambda)$ on $\psi_{p,\sigma}$

$$U(\Lambda)\psi_{p,\sigma} = U(\Lambda)\underbrace{U(L[p])\psi_{k,\sigma}}_{\text{definition of } \psi_{p,\sigma}} = \underbrace{U(\Lambda L[p])}_{\text{representation}}\ \psi_{k,\sigma}$$

$$= \underbrace{U(L[\Lambda p])U(L^{-1}[\Lambda p])}_{1} U(\Lambda L[p])\psi_{k,\sigma}$$

$$= U(L[\Lambda p])U(\underbrace{L^{-1}(\Lambda p)\Lambda L[p]}_{W(\Lambda,p)})\psi_{k,\sigma}$$

Wigner rotation:



- $W(\Lambda, p) \equiv$ Wigner rotation
  - Takes k back to itself, i.e. $W^{\mu}_{\ \nu} k^{\nu} = k^{\mu}$

Wigner Rotations and the Little Group

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How to classify different types of particles $\psi_{p,\sigma}$?

- Consider Lorentz transformation $U(\Lambda)$ on $\psi_{p,\sigma}$

$$U(\Lambda)\psi_{p,\sigma} = U(L[\Lambda p])U(\underbrace{L^{-1}[\Lambda p]\Lambda L[p]}_{W(\Lambda,p)})\psi_{k,\sigma}$$

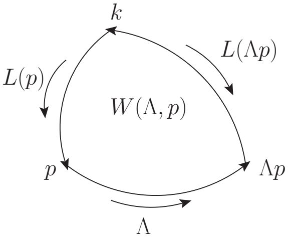
k unchanged

$$= U(L[\Lambda p])\sum_{\sigma'} D_{\sigma,\sigma'}(W[\Lambda,p])\psi_{k,\sigma'}$$

σ unchanged

$$= \sum_{\sigma'} D_{\sigma,\sigma'}(W[\Lambda,p])\psi_{\Lambda p,\sigma'}$$

Wigner rotation:



$W(\Lambda,p)$ leaves k invariant, i.e. $W^{\mu}_{\nu}k^{\nu} = k^{\mu}$

- Wigner rotation generates the '**little group**'
  - Subset of Lorentz transformation leaving k^{μ} invariant
  - Responsible for quantum numbers σ

Consequences for amplitudes

Lorentz transformation of a particle

$$U(\Lambda)\psi_{p,\sigma} = D_{\sigma,\sigma'}(W[\Lambda,p])\psi_{\Lambda p,\sigma'} \Leftrightarrow U(\Lambda)|p,\sigma\rangle = D_{\sigma,\sigma'}(W[\Lambda,p])|\Lambda p,\sigma'\rangle$$

- Scattering amplitudes are of the form $\mathcal{M}_n \sim \langle \text{particles out} | \text{particles in} \rangle$
- Each particle transforms under Lorentz/little groups

Lorentz transformation of an amplitude

$$\mathcal{M}_n^{\Lambda}(p_i,\sigma_i) \sim \prod_{i=1}^{n} D_{\sigma_i,\sigma_i'}(W[\Lambda,p_i])\mathcal{M}_n([\Lambda p]_i,\sigma_i')$$

- Can use the little group transformation to check consistency of an amplitude
  - Can sometimes be enough to fix the amplitude!!

Particle Classification

What are the different standard momenta $k^\mu = (E, k_x, k_y, k_z)$??
(Different k^μ cannot be (proper orthochronously) Lorentz transformed into one another)

Particle type	Standard momentum k	Little group
$p^2 = m^2, p^0 > 0$	$k^\mu = (m, 0, 0, 0)$	$\mathrm{SO}(3)$

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Particle type	Standard momentum k	Little group
$p^2 = m^2, p^0 > 0$	$k^\mu = (m, 0, 0, 0)$	$\mathrm{SO}(3)$
$p^2 = m^2, p^0 < 0$	$k^\mu = (-m, 0, 0, 0)$	$\mathrm{SO}(3)$
$p^2 = 0, p^0 > 0$	$k^\mu = (\omega, 0, 0, \omega)$	$\mathrm{ISO}(2)$
$p^2 = 0, p^0 < 0$	$k^\mu = (-\omega, 0, 0, \omega)$	$\mathrm{ISO}(2)$
$p^2 = -m^2$	$k^\mu = (0, 0, 0, m)$	$\mathrm{SO}(2, 1)$
$p^\mu = 0$	$k^\mu = (0, 0, 0, 0)$	$\mathrm{SO}(3, 1)$

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Particle Classification

What are the different standard momenta $k^\mu = (E, k_x, k_y, k_z)$??
(Different k^μ cannot be (proper orthochronously) Lorentz transformed into one another)

	Particle type	Standard momentum k	Little group
Massive	$p^2 = m^2, p^0 > 0$	$k^\mu = (m, 0, 0, 0)$	SO(3)
	$p^2 = m^2, p^0 < 0$	$k^\mu = (-m, 0, 0, 0)$	SO(3)
Massless	$p^2 = 0, p^0 > 0$	$k^\mu = (\omega, 0, 0, \omega)$	ISO(2)
	$p^2 = 0, p^0 < 0$	$k^\mu = (-\omega, 0, 0, \omega)$	ISO(2)
Tachyon	$p^2 = -m^2$	$k^\mu = (0, 0, 0, m)$	SO(2, 1)
Vacuum	$p^\mu = 0$	$k^\mu = (0, 0, 0, 0)$	SO(3, 1)

Not just $U(\Lambda)$ and p. Also important is k and little group quantum numbers!!

The $SO(3)$ Little Group of Massive Particles

Standard momentum $k^\mu = (m, 0, 0, 0)$
What are quantum numbers σ and reps $D(W)$??

- $D(W(\Lambda, p))$ representations of $so(3) \cong su(2)$
- This group is well known from non-relativistic quantum mechanics
- For particle of total spin j:
  - $\sigma = \{j^2 = j(j+1), j_s\}$ ($s \equiv$ some direction)
  - j^2 unchanged under Lorentz and little groups, j_s changes
- For infinitesimal rotations $R_{ik} = \delta_{ik} + \theta_{ik}$

$$D^{(j)}_{\sigma\sigma'}(1+\theta) = \delta_{\sigma\sigma'} + \frac{i}{2}\theta_{ik}(J^{(j)}_{ik})_{\sigma\sigma'}$$

$$\left(J^{(j)}_{23} \pm iJ^{(j)}_{31}\right)_{\sigma\sigma'} = \left(J^{(j)}_{1} \pm iJ^{(j)}_{2}\right)_{\sigma\sigma'} = \delta_{\sigma',\sigma\pm1}\sqrt{(j\mp\sigma)(j\pm\sigma+1)}$$

$$(J^{(j)}_{12})_{\sigma\sigma'} = (J^{(j)}_{3})_{\sigma\sigma'} = \sigma\delta_{\sigma'\sigma}$$

The $ISO(2)$ Little Group of Massless Particles

Standard momentum $k^\mu = (\omega, 0, 0, \omega)$
Quantum numbers are $\sigma = a, b, \theta$

It can be shown that $W^\mu{}_\nu$ is

$$W(a,b,\theta) = \underbrace{\begin{pmatrix} 1+\xi & a & b & -\xi \\ a & 1 & 0 & -a \\ b & 0 & 1 & -b \\ \xi & a & b & 1-\xi \end{pmatrix}}_{S(a,b)} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{R(\theta)}, \quad \xi = \frac{a^2+b^2}{2}$$

$$\xrightarrow{\text{infinitesimally}} \begin{pmatrix} 1 & a & b & 0 \\ a & 1 & -\theta & -a \\ b & \theta & 1 & -b \\ 0 & a & b & 1 \end{pmatrix} = 1 + ia\underbrace{(J_2 + K_1)}_{A} + ib\underbrace{(-J_1 + K_2)}_{B} + i\theta J_3$$

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Quantum Numbers of Massless Little Group $ISO(2)$

Standard momentum $k^\mu = (\omega, 0, 0, \omega)$
Little group generated by $W^\mu{}_\nu(a, b, \theta) = 1 + iaA + ibB + i\theta J_3$
Quantum numbers are $\sigma = a, b, \theta$
Commutation relations: $[J_3, A] = iB\ , \quad [J_3, B] = iA\ , \quad [A, B] = 0$

To calculate a, b, θ first diagonalise a, b:

- $A\psi_{k,a,b} = a\psi_{k,a,b}$
- $B\psi_{k,a,b} = b\psi_{k,a,b}$

Then define full state as:

- $\psi_{k,a,b,\theta} \equiv e^{-i\theta J_3}\psi_{k,a,b}$

Calculate eigenvalue of A on full state:

$$
\begin{aligned}
A\psi_{k,a,b,\theta} &= \underbrace{e^{-i\theta J_3} e^{i\theta J_3}}_{1} A \underbrace{e^{-i\theta J_3}\psi_{k,a,b}}_{\text{definition}} \\
&= e^{-i\theta J_3}(\underbrace{A\cos\theta - B\sin\theta}_{e^{i\theta J_3}Ae^{-i\theta J_3}})\psi_{k,a,b} \\
&= (a\cos\theta - b\sin\theta)\psi_{k,a,b,\theta}
\end{aligned}
$$

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Quantum Numbers of Massless Little Group $ISO(2)$

Standard momentum $k^\mu = (\omega, 0, 0, \omega)$
Little group generated by $W^\mu{}_\nu(a, b, \theta) = 1 + iaA + ibB + i\theta J_3$
Quantum numbers are $\sigma = a, b, \theta$
Commutation relations: $[J_3, A] = iB$, $[J_3, B] = iA$, $[A, B] = 0$

To calculate a, b, θ first diagonalise a, b:

- $A\psi_{k,a,b} = a\psi_{k,a,b}$
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Conclusion: Unless $a = b = 0$, $\theta \equiv$ continuous and $\exists$ infinity of states

Quantum Numbers of Massless Little Group $ISO(2)$

Standard momentum $k^\mu = (\omega, 0, 0, \omega)$
Little group generated by $W^\mu{}_\nu(a, b, \theta) = 1 + iaA + ibB + i\theta J_3$
Quantum numbers are either $\sigma = a, b, \theta$, with θ continuous or
Quantum numbers are $a = b = 0$ and $j_3 = \pm j =$ helicity

We don't see states with such a continuous parameter so two options:

1. Predict a new undiscovered type of particle
   - See e.g. hep-th:1302.1198, hep-th:1302.1577, hep-th:1302.3225, and hep-th:1404.0675
2. Assume $a = b = 0$ for physical states
   - Use topology to conclude that $j_3 \equiv J_3$ eigenvalue is quantised
     - $SL(2, \mathbb{C})$ is universal covering of $SO(2)$ so rotation by $4\pi \equiv$ identity
   - J_3 measures spin along motion $\equiv$ **helicity**

Conclusion: Only quantum number for massless is $\sigma = j_3 =$ helicity

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Massless particles

Standard momentum $k^\mu = (\omega, 0, 0, \omega)$
Physical particles: little group generated by $W^\mu{}_\nu(a, b, \theta) = 1 + i\theta J_3$
Quantum number is $\sigma = j_3 =$ helicity

Some notes:

- It is entirely natural to consider $h = +$ as a different particle to $h = -$
- If parity conserved: $\psi_{\vec{k},\sigma} \xrightarrow{\text{parity}} \psi_{-\vec{k},\sigma}$ so $h \to -h$
- $\Rightarrow$ consider $h = \pm$ as two reps of same particle
- Any massless particles (e.g. graviton, photon) have exactly two d.o.f
- No concept of total spin and spin along a direction of motion

(Quadratic) Casimirs of Poincaré Algebra

The two quadratic Casimirs of Poincaré algebra

$P^2 = m^2$ and $W^2 = -m^2 j(j+1) \overset{m=0}{=} -\omega^2(A^2+B^2)$

- $P_\mu = i\frac{\partial}{\partial x^\mu}$ generates translations
- $W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\lambda\omega}P_\nu J_{\lambda\omega}$ generates little group
- W^μ called Pauli-Lubanski pseudovector
- $(W^\mu/m)^2 = -j(j+1) \Rightarrow W^\mu/m$ is a covariant spin operator!

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Covariant (massive) spin-1/2 operator

Using Pauli-Lubanski (pseudo)vector

$$\frac{1}{2}\Sigma^\mu \equiv \frac{W^\mu}{m} = -\frac{1}{4m}\epsilon^{\mu\nu\lambda\omega}P_\nu\sigma_{\lambda\omega}\ , \qquad \sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] \overset{\text{spin}-1/2}{=} 2J^{\mu\nu}$$

- Check Σ^μ for the rest frame $P^\mu \to k^\mu = (m, 0, 0, 0)$

$$\frac{1}{2}\Sigma^i \overset{\text{rest}}{=} \frac{i}{4}\epsilon^{ijk}\gamma^j\gamma^k = \frac{1}{2}\gamma^5\gamma^0\gamma^i = \frac{1}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}\ , \qquad \frac{1}{2}\Sigma^0 \overset{\text{rest}}{=} 0\ ,$$

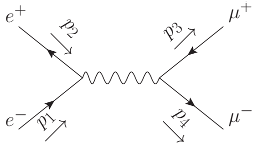
- We can measure Σ^μ in direction s^μ, $s^\mu \overset{\text{rest}}{=} (0, \hat{\mathbf{s}})$ using
$$-\frac{\Sigma^\mu s_\mu}{2} = \frac{1}{4m}\epsilon^{\mu\nu\lambda\omega}s_\mu P_\nu\sigma_{\lambda\omega}$$
- Above shows $s \cdot P = \Sigma \cdot P = 0$, $s^2 = -1$ and (with a bit of algebra)

$$-\frac{\Sigma^\mu s_\mu}{2} = \frac{1}{2m}\gamma^5 \not{s}\not{P} = \underbrace{\frac{1}{2}\gamma^5\not{s}}_{\not{P}\psi = m\psi} \equiv \text{operator measuring spin of spinor along } s^\mu$$

Spinor-Helicity Formalism

Traditional QFT: a Simple Example

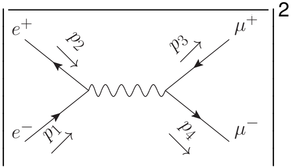
- Keep all particles unpolarised
- Obtain amplitude as matrix



$$\sim [\bar{v}_r(p_2)\gamma^\mu u_s(p_1)][\bar{u}_t(p_4)\gamma_\mu v_w(p_3)]$$

Traditional QFT: a Simple Example

- Keep all particles unpolarised
- Obtain amplitude as matrix
- Square the matrix amplitude
  - Spin states are orthogonal



$$\sim \sum_{r,s,t,w} [\bar{v}_r(p_2)\gamma^\mu u_s(p_1)][\bar{u}_t(p_4)\gamma_\mu v_w(p_3)]$$

$$\times [\bar{u}_s(p_1)\gamma^\nu v_r(p_2)][\bar{v}_w(p_3)\gamma_\nu u_t(p_4)]$$

Traditional QFT: a Simple Example

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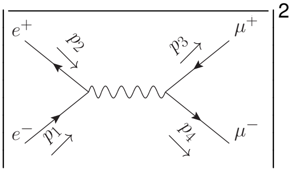
- Keep all particles unpolarised
- Obtain amplitude as matrix
- Square the matrix amplitude
  - Spin states are orthogonal
- Move components around



$$\sim \sum_{r,s,t,w} [\bar{v}_r(p_2)\gamma^\mu u_s(p_1)][\bar{u}_t(p_4)\gamma_\mu v_w(p_3)]$$

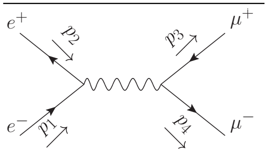
$$\times [\bar{u}_s(p_1)\gamma^\nu v_r(p_2)][\bar{v}_w(p_3)\gamma_\nu u_t(p_4)]$$

$$\sim \sum_{r,s,t,w} [\gamma^\nu v_r(p_2)\bar{v}_r(p_2)\gamma^\mu u_s(p_1)\bar{u}_s(p_1)]$$

$$\times [\gamma_\nu u_t(p_4)\bar{u}_t(p_4)\gamma_\mu v_w(p_3)\bar{v}_w(p_3)]$$

Traditional QFT: a Simple Example

- Keep all particles unpolarised
- Obtain amplitude as matrix
- Square the matrix amplitude
  - Spin states are orthogonal
- Move components around
- Use spin sums
- Take trace of fermionic structure
- Requires identities of γ^μ
- Simplify

$\Big|$  $\Big|^2$

$$\sim \mathrm{Tr}\left[\gamma^\nu(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\right]$$
$$\times\ \mathrm{Tr}\left[\gamma_\nu(\not{p}_4 + m_\mu)\gamma_\mu(\not{p}_3 + m_\mu)\right]$$

$$\mathrm{Tr}\left[\gamma^{\mu_1}\gamma^{\mu_2}\right] = 4g^{\mu_1\mu_2}$$
$$\mathrm{Tr}\left[\gamma^{\mu_1}\dots\gamma^{\mu_4}\right] =$$
$$4(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_3\mu_2})$$
$$\mathrm{Tr}\left[\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}\right] = 0$$

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The Helicity Basis: what and why?

Helicity basis means each particle has a specific helicity
Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles

- Ideal for (approximately) massless particles (e.g. most particles at LHC)
  - Helicity is the quantum number of the massless little group
  - For incoming (anti)spinors chirality $((\frac{1}{2},0)$ or $(0,\frac{1}{2})) \sim$ helicity $(-\frac{1}{2}$ or $+\frac{1}{2})$
  - For outgoing (anti)spinors chirality $((\frac{1}{2},0)$ or $(0,\frac{1}{2})) \sim$ helicity $(-\frac{1}{2}$ or $+\frac{1}{2})$
- Amplitude itself is a number rather than a matrix
  - Easy to square
- Different helicity amplitudes are orthogonal
  - Only sum over helicities after squaring

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Spinor-Helicity: its Building Blocks

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Spinors (use chiral basis):

$$u^+(p) = v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} \qquad u^-(p) = v^+(p) = \begin{pmatrix} |p] \\ 0 \end{pmatrix}$$

$$\bar{u}^+(p) = \bar{v}^-(p) = \begin{pmatrix} [p| & 0 \end{pmatrix} \qquad \bar{u}^-(p) = \bar{v}^+(p) = \begin{pmatrix} 0 & \langle p| \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_{L/R} = \frac{1 \mp \gamma^5}{2}$$

- Amplitude written in terms of Lorentz-invariant spinor inner products

$$\langle ij\rangle = -\langle ji\rangle \equiv \langle i||j\rangle \text{ and } [ij] = -[ji] \equiv [i||j]$$

  - These are well known complex numbers, $\langle ij\rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$
- Cannot contract left and right: $\langle i||j] \equiv \bar{u}(p_i)P_R P_L u(p_j) = 0$
  - Objects live in different Lorentz reps so a contraction makes no sense!

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Spinor-Helicity: Vectors and Removing μ Indices

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Dirac matrices in chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} \qquad \sqrt{2}\tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma}),$$

Remove $\tau/\bar{\tau}$ matrices in amplitude with

$$\underbrace{\langle i|\bar{\tau}^\mu|j][k|\tau_\mu|l\rangle = \langle il\rangle[kj]}_{\text{Fierz identity}}, \qquad \underbrace{\langle i|\bar{\tau}^\mu|j] = [j|\tau^\mu|i\rangle}_{\text{Charge Conjugation}}$$

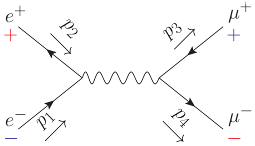
Express (massless) p^μ in terms of spinors

$$p^\mu = \frac{[p|\tau^\mu|p\rangle}{\sqrt{2}} = \frac{\langle p|\bar{\tau}^\mu|p]}{\sqrt{2}}, \quad \sqrt{2}p^\mu\tau_\mu \equiv \not{p} = |p]\langle p|, \quad \sqrt{2}p^\mu\bar{\tau}_\mu \equiv \bar{\not{p}} = |p\rangle[p|$$

Our Simple Example using Spinor Helicity

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

- Explicit helicities for external particles
- Now diagram is a complex number
  - Easy to square
- Square first, then sum over helicities
  - Some helicity configurations vanish
  - CP-invariance relates helicity configurations



$$\sim \langle p_2|\bar{\tau}^\mu|p_1]\langle p_4|\bar{\tau}_\mu|p_3]$$
$$= [p_1|\tau^\mu|p_2\rangle\langle p_4|\bar{\tau}_\mu|p_3]$$
$$= \langle p_4 p_2\rangle[p_1 p_3]$$

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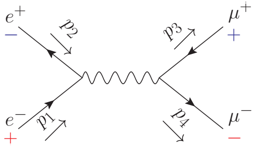

$$\sim [p_2|\tau^\mu|p_1\rangle\langle p_4|\bar{\tau}_\mu|p_3]$$
$$= \langle p_4 p_1\rangle [p_2 p_3]$$

Our Simple Example using Spinor Helicity

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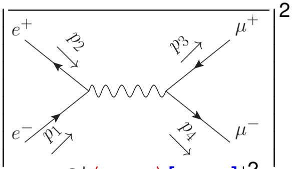

$$\sim 2|\langle p_4 p_2\rangle [p_1 p_3]|^2 + 2|\langle p_4 p_1\rangle [p_2 p_3]|^2$$

Spinor-Helicity: Gauge Bosons in Terms of Spinors

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Outgoing polarisation vectors:

$$\epsilon_+^\mu(p,r) = \frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp\rangle}\,, \qquad \epsilon_-^\mu(p,r) = \frac{[r|\tau^\mu|p\rangle}{[pr]}$$

$$p\cdot\epsilon_+(p,r) = \underbrace{\frac{\langle r|p^\mu\bar{\tau}_\mu|p]}{\langle rp\rangle} = 0}_{\text{Weyl eq. } p^\mu\bar{\tau}_\mu|p]=0} \qquad p\cdot\epsilon_-^\mu(p,r) = \underbrace{\frac{[r|p^\mu\tau_\mu|p\rangle}{[pr]} = 0}_{\text{Weyl eq. } p^\mu\tau_\mu|p\rangle=0}$$

$$\epsilon_+(p,r)\cdot(\epsilon_-)^*(p,r) = \underbrace{\frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp\rangle}\frac{[r|\tau_\mu|p\rangle}{[pr]}}_{\epsilon_\pm=(\epsilon_\mp)^*} = \frac{\langle rp\rangle[rp]}{\langle rp\rangle[pr]} = \underbrace{-1}_{[pr]=-[rp]}$$

Spinor-Helicity: Gauge Bosons in Terms of Spinors

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Outgoing polarisation vectors:

$$\epsilon_+^\mu(p,r) = \frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp\rangle}\,, \qquad \epsilon_-^\mu(p,r) = \frac{[r|\tau^\mu|p\rangle}{[pr]}$$

- r is a (massless) arbitrary reference momentum ($p \cdot r \neq 0$)
- Different r choices correspond to different gauges

$$\epsilon_+^\mu(p,r') - \epsilon_+^\mu(p,r) = -p^\mu \frac{\langle r' r\rangle}{\langle r' p\rangle\langle rp\rangle}$$

- Gauge invariant quantities must be r-invariant
  - Choose r as conveniently as possible (remember $\langle ij\rangle = -\langle ji\rangle$ s.t. $\langle ii\rangle = 0$)
  - Variance under $r \to r'$ good check of gauge invariance of (partial) amplitude

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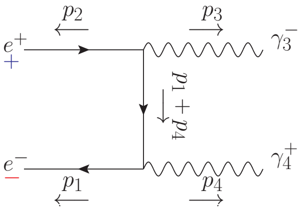
Another Spinor-Helicity Example: $e^+e^- \to \gamma\gamma$

Diagram 1:

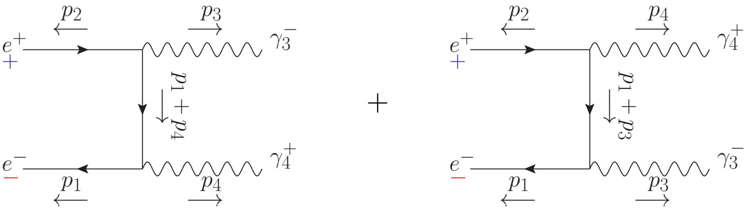


$$\sim \langle p_1|\bar{\tau}^\mu \underbrace{\left(|p_1]\langle p_1| + |p_4]\langle p_4|\right)}_{\not{p}_1+\not{p}_4} \bar{\tau}^\nu |p_2] \underbrace{\frac{\langle r_3|\bar{\tau}_\nu|p_3]}{\langle r_3 3\rangle}}_{\epsilon_3^-} \underbrace{\frac{[r_4|\tau_\mu|p_4\rangle}{[p_4 4]}}_{\epsilon_4^+}$$

$$= \underbrace{\frac{\langle 1 r_4\rangle\left([41]\langle 13\rangle + [44]\langle 43\rangle\right)[r_3 2]}{\langle r_3 3\rangle[4 r_4]}}_{\text{Fierz identities like } \langle i|\bar{\tau}^\mu|j][k|\tau_\mu|l\rangle = \langle il\rangle[kj]} = \underbrace{\frac{\langle 1 r_4\rangle[41]\langle 13\rangle[r_3 2]}{\langle r_3 3\rangle[4 r_4]}}_{[ii]=0}$$

Another Spinor-Helicity Example $e^+e^- \to \gamma\gamma$

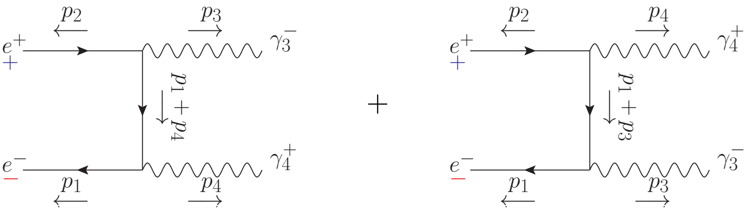
Add two diagrams together



$$\sim \frac{\langle 1 r_4\rangle[41]\langle 13\rangle[r_3 2]}{\langle r_3 p_3\rangle[p_4 r_4]} + \frac{\langle 13\rangle([r_3 1]\langle 1 r_4\rangle + [r_3 3]\langle 3 r_4\rangle)[42]}{\langle r_3 3\rangle[4 r_4]}$$

Another Spinor-Helicity Example $e^+e^- \to \gamma\gamma$

Choose gauge d.o.f. wisely ($r_4 = p_1$, $r_3 = p_2$ s.t. $\langle 1 r_4\rangle = [r_3 2] = 0$)
Recall: $\langle ii\rangle = [jj] = 0$ due to antisymmetry



$$\sim \frac{\langle 1 r_4\rangle[41]\langle 13\rangle[r_3 2]}{\langle r_3 3\rangle[4 r_4]} + \frac{\langle 13\rangle([r_3 1]\langle 1 r_4\rangle + [r_3 3]\langle 3 r_4\rangle)[42]}{\langle r_3 3\rangle[4 r_4]}$$

$$= 0 + \frac{\langle 13\rangle[23]\langle 31\rangle[42]}{\langle 23\rangle[41]}$$

Spinor-Helicity and the Little Group

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Little Group and Spinor Helicity

Little Group

Group of transformations that leaves p^μ invariant

- Recall: $\not{p} = |p]\langle p|$ and $\bar{\not{p}} = |p\rangle[p|$
- $\Rightarrow$ under little group transformation:
  - $\langle p| \to t\langle p|$ and $|p\rangle \to t|p\rangle$
  - $|p] \to t^{-1}|p]$ and $[p| \to t^{-1}[p|$
  - $p \in \mathbb{R} \Rightarrow t = e^{i\theta/2} = e^{i\theta|h|}$
  - $p \in \mathbb{C} \Rightarrow t$ more general
- Recall: $\epsilon_+^\mu(p,r) = \frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp\rangle}$ and $\epsilon_-^\mu(p,r) = \frac{[r|\tau^\mu|p\rangle}{[pr]}$ (p outgoing)
- $\Rightarrow$ under little group transformation:
  - $\epsilon_+^\mu \to t^{-2}\epsilon_+^\mu$
  - $\epsilon_-^\mu \to t^2\epsilon_-^\mu$

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Little Group and Three-Point Amplitudes

Transform (outgoing) particle i under little group

$$\mathcal{M}(|i\rangle, |i], h_i) \to \mathcal{M}(t|i\rangle, t^{-1}|i], h_i) = t^{-2h_i}\mathcal{M}(|i\rangle, |i], h_i)$$

- Ansatz: An amplitude can be written either entirely in terms of $\langle ij\rangle$ or $[ij]$
- $\Rightarrow$ either:
  - $\mathcal{M}(1^{h_1}, 2^{h_2}, 3^{h_3}) = c_{\text{angle}}\langle 12\rangle^{x_{12}}\langle 23\rangle^{x_{23}}\langle 31\rangle^{x_{31}}$ or
  - $\mathcal{M}(1^{h_1}, 2^{h_2}, 3^{h_3}) = c_{\text{square}}[12]^{y_{12}}[23]^{y_{23}}[31]^{y_{31}}$

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  - $\mathcal{M}(1^{h_1}, 2^{h_2}, 3^{h_3}) = c_{\text{square}}[12]^{y_{12}}[23]^{y_{23}}[31]^{y_{31}}$
- Scale particle 1: $\Rightarrow$ either:
  - $\mathcal{M} \to t^{-2h_1}\mathcal{M} = t^{x_{12}+x_{31}}\mathcal{M} \Rightarrow -2h_1 = x_{12} + x_{31}$ or
  - $\mathcal{M} \to t^{-2h_1}\mathcal{M} = t^{-y_{12}-y_{31}}\mathcal{M} \Rightarrow 2h_1 = y_{12} + y_{31}$

Little Group and Three-Point Amplitudes

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  - $\mathcal{M} \to t^{-2h_1}\mathcal{M} = t^{-y_{12}-y_{31}}\mathcal{M} \Rightarrow 2h_1 = y_{12} + y_{31}$
- Solving for all particles gives:
  - $\mathcal{M}(1^{h_1}, 2^{h_2}, 3^{h_3}) = c_{\text{angle}}\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_1-h_3}$ or
  - $\mathcal{M}(1^{h_1}, 2^{h_2}, 3^{h_3}) = c_{\text{square}}[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_1+h_3-h_2}$

Little Group and Three-Point Amplitudes

Three-point amplitude possible solutions

$$\mathcal{M}(1^{h_1}, 2^{h_2}, 3^{h_3}) = c_{\text{angle}} \langle 12\rangle^{h_3-h_1-h_2} \langle 23\rangle^{h_1-h_2-h_3} \langle 31\rangle^{h_2-h_1-h_3} \text{ or}$$
$$\mathcal{M}(1^{h_1}, 2^{h_2}, 3^{h_3}) = c_{\text{square}} [12]^{h_1+h_2-h_3} [23]^{h_2+h_3-h_1} [31]^{h_1+h_3-h_2}$$

- Which of our two solutions to choose??
- Use mass dimension: $[\langle ij\rangle] = [[ij]] = [p]$ and $[\mathcal{M}_n] = [p]^{4-n}$
  - Three-point amplitudes $\mathcal{M}_3$ have $[\mathcal{M}_3] = [p]$
  - Choose whichever option gives correct mass dimension of coupling c

Little Group and Three-Point Amplitudes

Three-point amplitude possible solutions

$\mathcal{M}(1^{h_1}, 2^{h_2}, 3^{h_3}) = c_{\text{angle}} \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_1-h_3}$ or
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  - Choose whichever option gives correct mass dimension of coupling c

Example: $h_1 = -h_2 = -h_3 = 1$ (e.g. three-gluon amplitude)

- $\mathcal{M}(1^+, 2^-, 3^-) = c_{\text{angle}} \langle 12 \rangle^{-1} \langle 23 \rangle^{3} \langle 31 \rangle^{-1} \Rightarrow c_{\text{angle}}$ dimensionless
- $\mathcal{M}(1^+, 2^-, 3^-) = c_{\text{square}} [12]^{1} [23]^{-3} [31]^{1} \Rightarrow c_{\text{square}}$ has dim 2

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Little Group and Three-Point Amplitudes

Three-point amplitude possible solutions

$$\mathcal{M}(1^{h_1},2^{h_2},3^{h_3}) = c_{\text{angle}}\langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_1-h_3} \text{ or}$$
$$\mathcal{M}(1^{h_1},2^{h_2},3^{h_3}) = c_{\text{square}}[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_1+h_3-h_2}$$

- Which of our two solutions to choose??
- Use mass dimension: $[\langle ij\rangle] = [[ij]] = [p]$ and $[\mathcal{M}_n] = [p]^{4-n}$
  - Three-point amplitudes $\mathcal{M}_3$ have $[\mathcal{M}_3] = [p]$
  - Choose whichever option gives correct mass dimension of coupling c

Example: $h_1 = -h_2 = -h_3 = 1$ (e.g. three-gluon amplitude) Correct 3-gluon amp

- $\mathcal{M}(1^+,2^-,3^-) = c_{\text{angle}}\langle 12\rangle^{-1}\langle 23\rangle^{3}\langle 31\rangle^{-1} \Rightarrow c_{\text{angle}}$ dimensionless
- $\mathcal{M}(1^+,2^-,3^-) = c_{\text{square}}[12]^{1}[23]^{-3}[31]^{1} \Rightarrow c_{\text{square}}$ has dim 2

Three-point amplitudes completely fixed by little group!

Note: Requires complex momenta for non-zero amplitude

BCFW Recursion and the MHV amplitudes

Recall: three-point amplitudes completely fixed by little group

Basic (oversimplified) idea of BCFW:

- Take known compact form of n-point amplitude
- Sum over all possible three-point amplitude attachments
- Write down compact form of $(n+1)$-point amplitude
- Recurse

Example: MHV (Maximally Helicity Violating) amplitude for n-gluon scattering

$$\mathcal{M}_n(1^-, 2^-, 3^+, \cdots, n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}$$

(see hep-th:0501052 and hep-ph:1308.1697 for BCFW details)

Game Time: Guess the Theory from the Amplitude

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Required Knowledge

$\mathcal{M}(|i\rangle, |i], h_i) \xrightarrow{\text{little group}} \mathcal{M}(t|i\rangle, t^{-1}|i], h_i) = t^{-2h_i}\mathcal{M}(|i\rangle, |i], h_i)$

$[\mathcal{M}_n] = [p]^{4-n}$ and $[\langle ij\rangle] = [[ij]] = [p]$

All particles outgoing

Questions: (i) What are helicities? (ii) What dimension is coupling? (iii) What theory?

Amplitude 1: $\mathcal{M}_5 = g_1 \frac{[13]^4}{[12][23][34][45][51]}$

Amplitude 2: $\mathcal{M}_4 = g_2 \frac{\langle 14\rangle\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle}$

Amplitude 3: $\mathcal{M}_4 = g_3 \frac{\langle 12\rangle^7[12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2}$

Game taken from section 2.6 of hep-ph:1308.1697

Game Time: Answer to First Amplitude

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Required Knowledge

$\mathcal{M}(|i\rangle, |i], h_i) \xrightarrow{\text{little group}} \mathcal{M}(t|i\rangle, t^{-1}|i], h_i) = t^{-2h_i}\mathcal{M}(|i\rangle, |i], h_i)$

$[\mathcal{M}_n] = [p]^{4-n}$ and $[\langle ij\rangle] = [[ij]] = [p]$

All particles outgoing

Questions: (i) What are helicities? (ii) What dimension is coupling? (iii) What theory?

Amplitude 1: $\mathcal{M}_5 = g_1 \frac{[13]^4}{[12][23][34][45][51]}$

(i) E.g. particle 1 under little group: $\mathcal{M}_5 \to \frac{t_1^{-4}}{t_1^{-2}}\mathcal{M}_5 \Rightarrow h_1 = 1$

All particles: $h_1 = h_3 = -h_2 = -h_4 = -h_5 = 1$

(ii) $[\mathcal{M}_5] = -1 = \left[\frac{[13]^4}{[12][23][34][45][51]}\right] \Rightarrow [g_2] = 0$

(iii) Yang Mills (spin-1 massless particles interacting)

Game Time: Answers

Required Knowledge

$\mathcal{M}(|i\rangle, |i], h_i) \xrightarrow{\text{little group}} \mathcal{M}(t|i\rangle, t^{-1}|i], h_i) = t^{-2h_i}\mathcal{M}(|i\rangle, |i], h_i)$

$[\mathcal{M}_n] = [p]^{4-n}$ and $[\langle ij\rangle] = [[ij]] = [p]$

All particles outgoing

Questions: (i) What are helicities? (ii) What dimension is coupling? (iii) What theory?

Amplitude 1: $\mathcal{M}_5 = g_1 \frac{[13]^4}{[12][23][34][45][51]}$

(i) $h_1 = h_3 = -h_2 = -h_4 = -h_5 = 1$ (ii) dimensionless (iii) Yang Mills

Amplitude 2: $\mathcal{M}_4 = g_2 \frac{\langle 14\rangle\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle}$

(i) $h_1 = h_2 = 0$ and $h_3 = -h_4 = 1$ (ii) dimensionless (iii) Scalar QED/QCD

Amplitude 3: $\mathcal{M}_4 = g_3 \frac{\langle 12\rangle^7[12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2}$

(i) $h_1 = h_2 = -h_3 = -h_4 = -2$ (ii) dim -2 (iii) Effective gravity

Conclusions

- Particles classified by their representation under *both* Lorentz and little groups
- Massive particles have total spin and spin along a given direction
- Massless particles only have helicity
- Spinor-helicity formalism simplifies amplitude calculations since amplitude is a complex number
- Using both spinor-helicity and the little group recursive amplitude calculations possible
  - These skip Feynman diagram step
  - Far more efficient