Projective Geometry and Amplitudes

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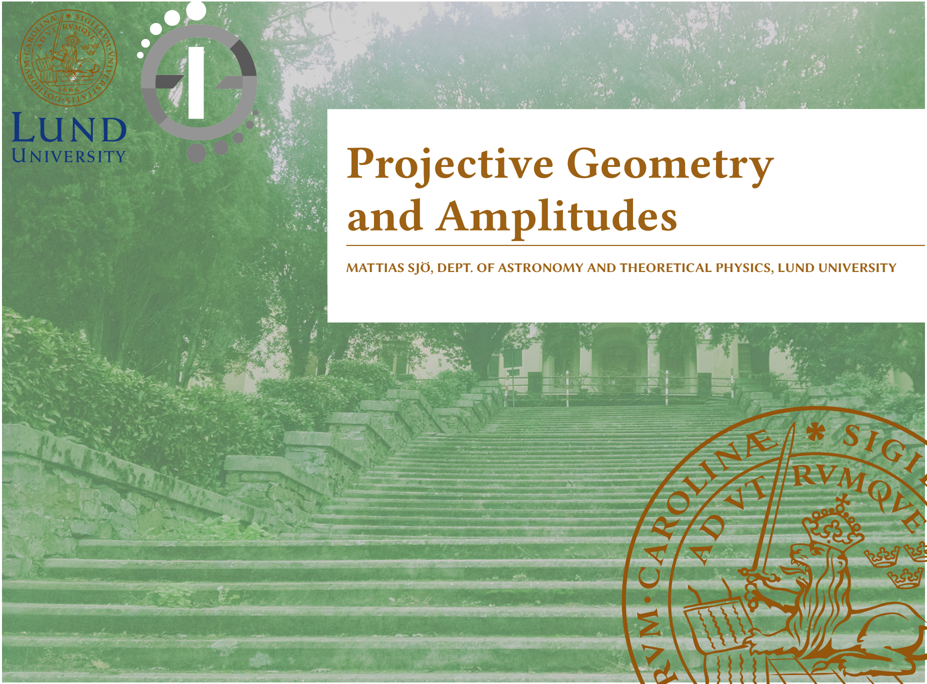


Background

The Galileo Galilei Institute

- Florence, Italy, near Galileo's home
- Hosts conferences, workshops and schools
- Torbjörn and I attended the Winter School 2020

Nima Arkani-Hamed

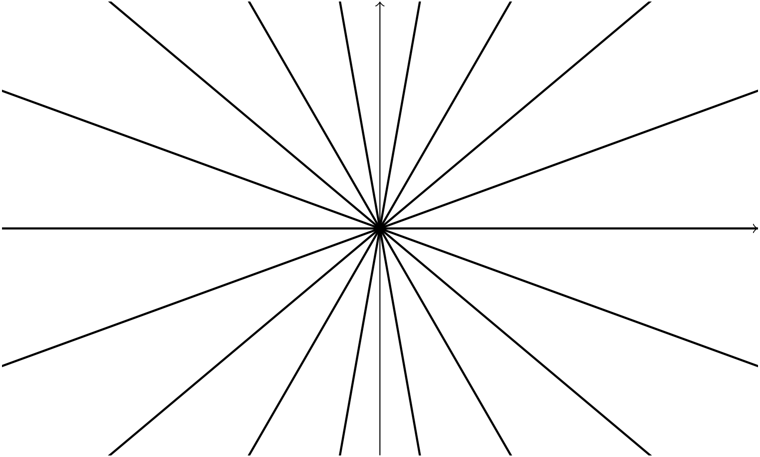
- Theoretical physicist at Princeton IAS
- Lectures "Positive Geometry of the Real World"
- I will present a subset of this topic

Projective space: artist's impression

Image: Getty Images

Definition of projective space $\mathbb{P}^n$

- Points in $\mathbb{P}^n \equiv$ Lines through origin in $\mathbb{R}^{n+1}$
- Alternatively: Equivalence classes in $\mathbb{R}^{n+1}$ under rescaling $\vec{r} \to \lambda\vec{r}, \lambda \neq 0$

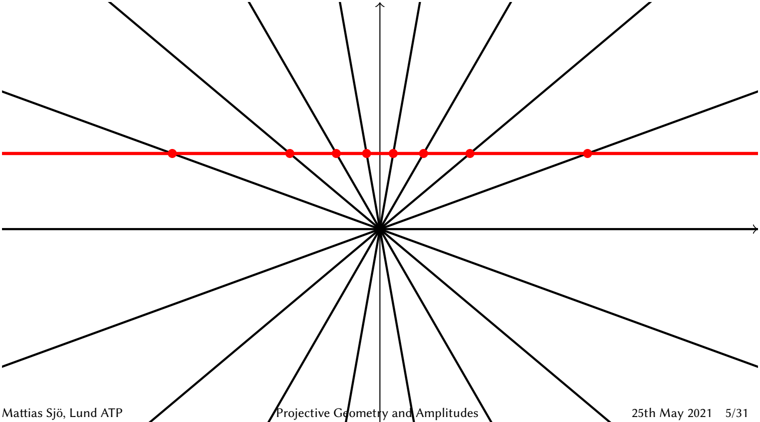


(Homogeneous) coordinates in $\mathbb{P}^n$

Fix coordinates as intersection with hyperplane:

$$X^I = \begin{pmatrix} 1 \\ \vec{z} \end{pmatrix}, \qquad \vec{z} = \text{position on hyperplane}$$

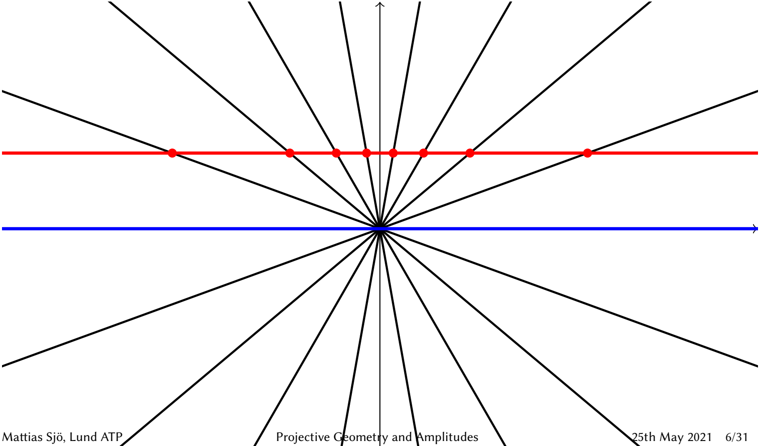
(up to rescaling)



Points at infinity

Some lines (exactly one in $\mathbb{R}^2$) do not intersect the hyperplane — These are the points at infinity:

$$X^I_\infty = \begin{pmatrix} 0 \\ \vec{z} \end{pmatrix}, \qquad \vec{z} = \text{orientation}$$



Lines in the projective plane $\mathbb{P}^2$

- General line in Euclidean plane:

$$0 = a + bx + cy$$

- "Projectivise":

$$X = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}, \qquad L = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ b/a \\ c/a \end{pmatrix}$$

$$0 = L_0 X^0 + L_1 X^1 + L_2 X^2 \equiv L_I X^I \quad \text{(Einstein summation)}$$

- Line at infinity:

$$L_I^\infty = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \quad \Rightarrow \quad L_I^\infty X_\infty^I = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vec{z} \end{pmatrix} = 0$$

Coordinate transformations

- Most general coordinate change on $\mathbb{P}^n$:

$$\mathrm{GL}(n+1)$$

- Tensor notation ($\Lambda \in \mathrm{GL}(n+1)$):

$$X^{I'} = \Lambda^{I'}{}_{I} X^{I}, \qquad L_{I'} = (\Lambda^{-1})_{I'}{}^{I} L_{I}$$

- With $X = \binom{1}{\vec{z}}$: Acts on $\vec{z}$ like

$$\vec{z} \to \frac{\vec{c} + D\vec{z}}{a + \vec{b} \cdot \vec{z}}, \qquad \Lambda = \begin{pmatrix} a & \vec{b}^{\mathrm{T}} \\ \vec{c} & D \end{pmatrix}$$

Most general nonlinear transformation that preserves straight lines (naïve expectation would be $\mathrm{GL}(n)$ plus translations)

The Levi-Civita tensor: points

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There can only be one invariant 3-index object:

$$\epsilon_{I\mathcal{J}K} \qquad \text{(totally antisymmetric)}$$

For brevity and convenience:

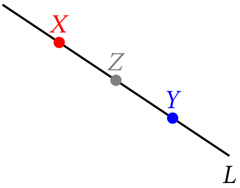
$$\epsilon_{I\mathcal{J}K} X^I Y^{\mathcal{J}} Z^K \equiv \langle XYZ \rangle$$

- Line L determined by two points X, Y

$$\Rightarrow \quad L_I = \epsilon_{I\mathcal{J}K} X^{\mathcal{J}} Y^K$$

- So X, Y, Z are collinear if

$$0 = \langle XYZ \rangle$$



The Levi-Civita tensor: lines

Upper-index version is equivalent:

$$\epsilon_{IJK} = \epsilon^{IJK}$$

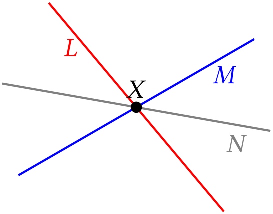
Similarly,

$$\epsilon^{IJK} L_I M_J N_K \equiv \langle LMN \rangle.$$

- Point X intersection of lines L, M

$$\Rightarrow \quad X^I = \epsilon^{IJK} L_J M_K$$

(Might be X_∞ if lines are parallel)



- Three lines L, M, N intersect in a single point if

$$0 = \langle LMN \rangle$$

The “Golden Rule” of $\mathbb{P}^n$

The only *valid* thing is also *correct*!

“I hope you never do geometry the same way again”
— Nima Arkani-Hamed

Conics in $\mathbb{P}^2$

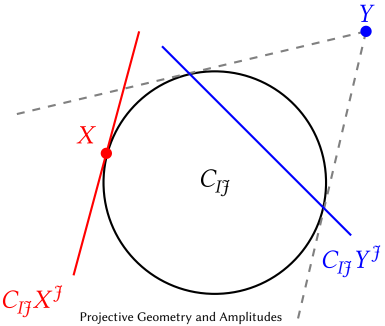
- General conic in Euclidean plane:

$$0 = a + bx + cy + dx^2 + exy + fy^2$$

- "Projectivise":

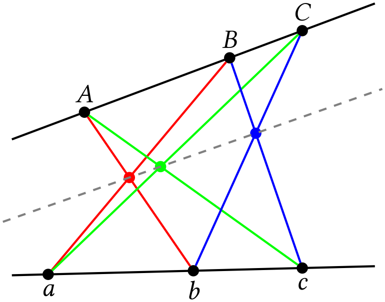
$$0 = C_{00}X^0X^0 + 2C_{01}X^0X^1 + C_{11}X^1X^1 + \ldots \equiv C_{IJ}X^IX^J$$

  with C_{IJ} symmetric 2-tensor.
- What is $C_{IJ}X^J$? — It's a line!
  (The only unique line that makes sense)



Pappus' Theorem

> "The first nontrivial result ever in mathematics"
>
> — David Hilbert



- Quite difficult to prove in $\mathbb{R}^2$
- Straightforward in $\mathbb{P}^2$: just use above identities with

$$\epsilon_{IJK}\epsilon^{KLM} = \delta_I^L\delta_J^M - \delta_I^M\delta_J^L$$

Canonical differential forms

Image: Wikipedia

Differential forms and their poles

What is a differential form?

Expression that can be integrated, like

$$f(x)\,\mathrm{d}x, \qquad g(x,y)\,\mathrm{d}x\,\mathrm{d}y, \qquad \ldots$$

(multiple differentials combined with wedge product)

Simple pole at $z = 0$ if proportional to

$$\frac{\mathrm{d}z}{z}$$

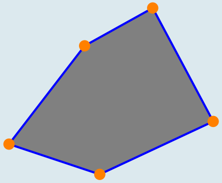
(other variants can be removed by coordinate redefinitions)

Polytopes

What is a polytope?

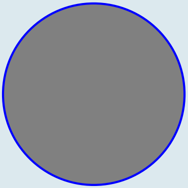
- Generalised polygon, polyhedron,...
- Object with boundaries of all codimensions
  Codim = (dim of space) − (dim of object)

2D polytope



Codim-1 boundary
and Codim-2 boundary

Not a 2D polytope



Only Codim-1 boundary

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Canonical forms

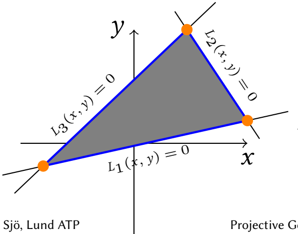
What is a canonical form?

- D-dimensional differential form Ω
- Poles on polytope boundary but not elsewhere
- Unique (up to scale) for each D-dim. polytope

- 1D: canonical form on interval $[a, b]$

$$\rightarrow \quad \Omega = \frac{\mathrm{d}x}{(x-a)(x-b)}$$

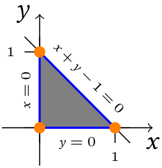
- 2D: canonical form on triangle bounded by $L_{1,2,3}(x, y) = 0$:



$$\rightarrow \quad \Omega = \frac{\mathrm{d}x\,\mathrm{d}y}{L_1(x, y) L_2(x, y) L_3(x, y)}$$

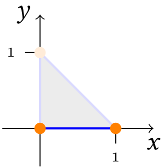
A closer look at the poles

- Simple example:



$$\rightarrow \quad \Omega = \frac{\mathrm{d}x\,\mathrm{d}y}{xy(1-x-y)}$$

- Residue at $y = 0$ (sans scale):



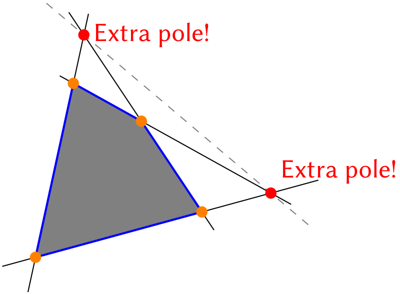
$$\rightarrow \quad \Omega|_{y=0} = \frac{\mathrm{d}x}{x(1-x)}$$

Canonical form of the codim-1 boundary $[0, 1]$!

- In general: N-order pole $\Leftrightarrow$ codim-N boundary

Numerator functions

- Problem with non-simplical polytopes:



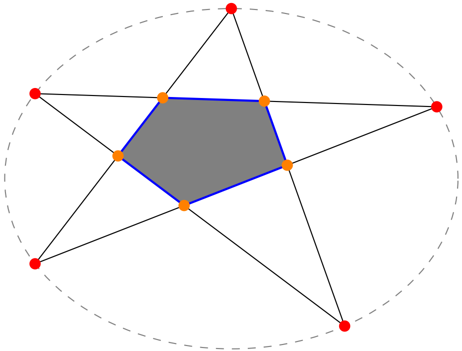
- Solution:

$$\{N(x,y)=0\} = \text{unique line through extra poles}$$

$$\Rightarrow \quad \Omega = \frac{N(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\prod L_i(x,y)}$$

- Must be line — higher-order function gives pole at infinity!

Numerator functions



$\{N(x, y) = 0\}$ = unique conic through extra poles

The projective approach

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Basic differential form in $\mathbb{P}^n$

$$\mathrm{d}z_1\,\mathrm{d}z_2\ldots\mathrm{d}z_n = \left\langle X\ \underbrace{\mathrm{d}X\,\mathrm{d}X\cdots\mathrm{d}X}_{n\text{ times}}\right\rangle$$

- Want to make rescaling-invariant object ($\mathbb{P}^1$ at first):

$$\Omega = \frac{\langle X\mathrm{d}X\rangle\langle AB\rangle}{\langle XA\rangle\langle XB\rangle}$$

- Let

$$X = \begin{pmatrix}1\\ z\end{pmatrix}, \qquad A = \begin{pmatrix}1\\ a\end{pmatrix}, \qquad B = \begin{pmatrix}1\\ b\end{pmatrix}$$

$$\Rightarrow \qquad \Omega = \frac{(b-a)\,\mathrm{d}z}{(z-a)(z-b)}$$

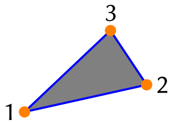
Canonical form on $[a, b]$ — even normalised!

The projective approach

Remember the “golden rule”:

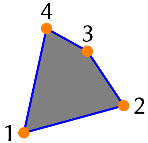
The only *valid* thing is also *correct*!

- Just write down unique rescaling-invariant form:



$$\rightarrow \quad \Omega = \frac{\langle X\,\mathrm{d}X\,\mathrm{d}X\rangle\langle 123\rangle^2}{\langle X12\rangle\langle X23\rangle\langle X31\rangle}$$

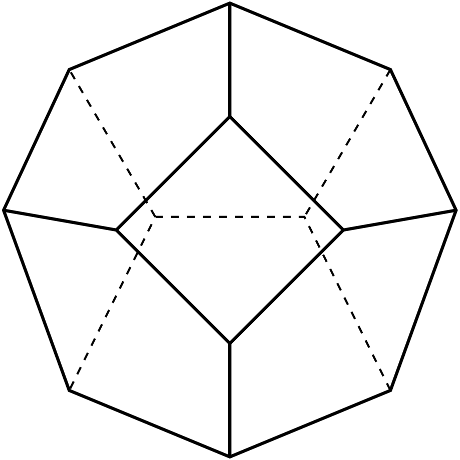
- Numerator functions “just fall out”:



$$\rightarrow \quad \Omega = \frac{\langle X\,\mathrm{d}X\,\mathrm{d}X\rangle L_I X^I}{\langle X12\rangle\langle X23\rangle\langle X34\rangle\langle X41\rangle},$$

$L_I =$ straightforward unique combination of $1, 2, 3, 4$

Scattering amplitudes



Theories with ordered amplitudes

- Common phenomenon:

$$\mathcal{M}(\underbrace{p_1,\dots,p_n}_{\text{momenta}};\underbrace{a_1,\dots,a_n}_{\text{colour/flavour indices}}) = \underbrace{\sum_{\sigma}}_{\text{permutations}} \mathcal{M}_\sigma(p_1,\dots,p_n)\mathcal{A}_\sigma(a_1,\dots,a_n)$$

where

$$\mathcal{M}_\sigma(p_1,\dots,p_n) = \mathcal{M}_{\text{id}}(p_{\sigma_1},\dots,p_{\sigma_n})$$

- Examples:

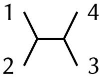
$$\begin{cases} \text{Colour-ordering} & \begin{cases}\text{Quantum Chromodynamics}\\ \text{Super Yang-Mills}\end{cases} \\ \text{Flavour-ordering} & \begin{cases}\text{Chiral Perturbation Theory}\\ \text{Various EFTs}\end{cases} \\ \text{Toy ordering} & \text{bi-adjoint } \phi^3\text{: } \mathcal{L} = f_{ABC}f^{abc}\phi_a^A\phi_b^B\phi_c^C \end{cases}$$

Ordered Feynman diagrams

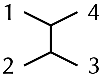
Why ordered diagrams are useful:

Only need to consider planar diagrams with external legs in order

- 4-point in bi-adjoint ϕ^3:



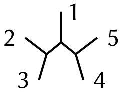
(s-channel) $+$



(t-channel)

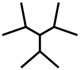
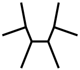
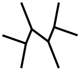
No u-channel! (non-planar or non-ordered)

- 5-point:



$+$ (4 cyclic permutations)

- 6-point:

 $+$  $+$  $+$  $+$ (cycl. perm.)

Amplitudes as differential forms

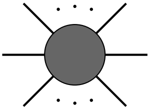
Why we do all this

$$\sigma(\text{process}) = \int \underbrace{|\mathcal{M}|^2 \, \mathrm{d}(\text{kinematics})}_{\text{differential form!}}$$

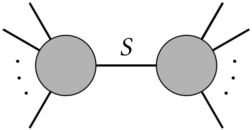
- *Ordered* amplitude is canonical form of some polytope in kinematic space
- Therefore: Makes sense to study its poles!

Factorisation of amplitudes

- Amplitude: n-point function with external legs on-shell

$\mathcal{M} =$ 

- (simple) pole whenever internal propagator goes on-shell

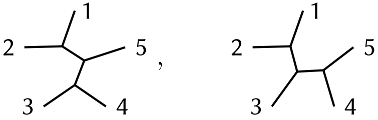
$\mathcal{M} \longrightarrow$ 

$= \dfrac{\mathcal{M}_L \mathcal{M}_R}{S - m^2}$

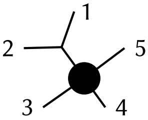
Each half is also an amplitude!

Factorisation of amplitudes

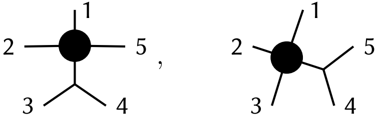
- These diagrams share the pole $S = (p_1 + p_2)^2$:



- That is, both are factorisations of

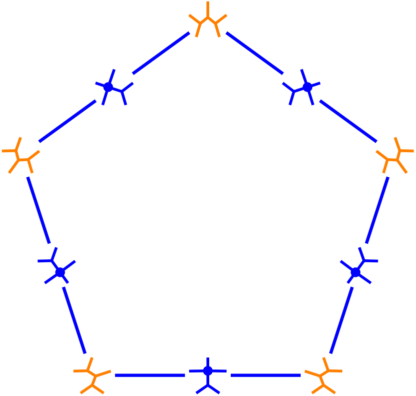


- But they are also factorisations of, respectively,



The 5-point associahedron

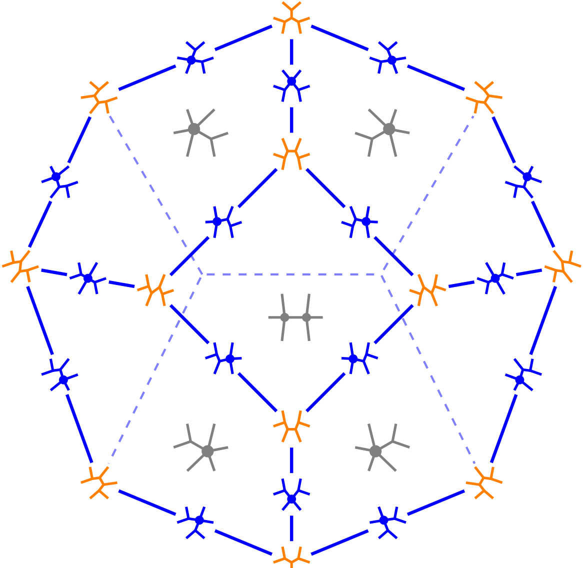
Map out the factorisation relationship:



Edge = simple pole   Vertex = double pole

Just like the canonical form!

The 6-point associahedron



Final remarks

- Diagrams $\rightarrow$ Associahedron $\rightarrow$ Projective Space $\rightarrow$ Canonical Form $\rightarrow$ Amplitude
- Requires explicit shape of associahedron. Obtained through beautiful kinematic manipulations — but not enough time!
- Similar treatment of Super Yang-Mills yields the famous Amplituhedron
- Projective space, canonical forms and associahedra have many applications unrelated to this

> "We declare as *interesting* only those problems that have sufficiently simple solutions, and declare as *engineering* those that don't."
>
> — Nima Arkani-Hamed

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