Making the colour matrix sparse at next-to-leading colour

Timea Vitos
based on a work with Rikkert Frederix

Lund University

Science Coffee,
September 7 2021

$$\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \longrightarrow \begin{pmatrix} x & 0 & x \\ 0 & 0 & 0 \\ x & 0 & x \end{pmatrix}$$

QCD in the Standard Model: some group theory

- Recall SM gauge group:

$$\mathrm{SU}(3)_C \times \mathrm{SU}(2)_w \times \mathrm{U}(1)_Y \to \mathrm{SU}(3)_C \times \mathrm{U}(1)_{\mathrm{QED}} \tag{1}$$

- Colour group most "complicated" in automated calculations
- Group transformation under local $\mathrm{SU}(3)_C$

$$\Psi(x) \to \underbrace{e^{-i\theta_a(x)T^a}}_{U \in \mathrm{SU}(3)_C} \Psi(x) \tag{2}$$

with $N_c^2 - 1 = 8$ traceless Hermitian generators T^a

$$[T^a, T^b] = i f^{abc} T^c \tag{3}$$

- Structure constants $f^{abc} \neq 0$ for non-abelian theory
- Adjoint representation: $(F^a)_{bc} = f^{abc}$

The U(1) part of the gluon

- Quark carry a $\mathbf{N_c}$ index i, anti-quarks a $\overline{\mathbf{N}}_\mathbf{C}$ index j
- Gluons carry a $\mathbf{N_c} \otimes \overline{\mathbf{N}}_\mathbf{C}$ index (i,j) $(\mathbf{N_c} \otimes \overline{\mathbf{N}}_\mathbf{C} = 1 + (\mathbf{N_c}^2 - 1)\ )$

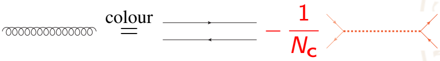
$$A_\mu^a \to A^i_{\mu j} \tag{4}$$

with $A^i_{\mu i} = 0$ (tracelessness)

- The Fierz identity captures this relation:

$$(T^a)_{ij}(T^a)_{kl} = T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl} \right) \tag{5}$$

- The gluon propagator becomes



(6)

is decomposed into a U(N_c) part and a U(1) part

- **The U(1) couples only to quarks!**

The large-N_c limit

- An attempt to make non-perturbative QCD perturbative
- First introduced by Gerard ‘t Hooft (1974) [1]
- Use the model

$$SU(3)_C \to SU(N_c) \tag{7}$$

and then $N_c \to \infty$

[1] G. ‘t Hooft. Nucl. Phys. B72 (1974) 461 - 473

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- Running of the strong $\alpha_S = \frac{g^2}{4\pi}$ coupling

$$\alpha_S(Q) = \frac{\alpha_S(\mu^2)}{1 + b_0 \frac{\alpha_S(\mu^2)}{4\pi} \log \frac{Q^2}{\mu^2}} \quad , \quad b_0 = \frac{11}{3} N_C - \frac{2}{3} N_F \tag{8}$$

if one defines $g \to \frac{g}{\sqrt{N_C}}$, same RGE is recovered

- Fix $g^2 N_C$ while taking the large-N_C limit

[1] G. 't Hooft. Nucl. Phys. B72 (1974) 461 - 473

The N_c^{-1} expansion

- In the limit $N_c \to \infty$, the U(1) part of the gluon vanishes
- Observables are expanded in terms of $\frac{1}{N_c}$ $\to$ colour expansion
- In the SM, $N_c = 3$: expansion in ~ 0.3

☹

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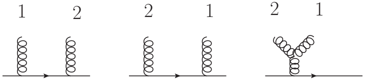
- In the SM, $N_c = 3$: expansion in ~ 0.3

- Effectively, an expansion in N_c^{-2} $\to \sim 10\%$ accuracy at order $(1/N_c)^2$!

Colour decompositions of amplitudes

- Feynman diagrams: Factorize the $SU(N_c)$ colour factor and the rest
- As an example, the $q\overline{q}+2g$ amplitude has contributions from



- Consider all meaningful permutations of external particles:

$$\mathcal{M} = g^n \sum_{\sigma \in S} \underbrace{C(\sigma)}_{\text{colour factor}} \times \underbrace{\mathcal{A}(\sigma)}_{\text{dual amplitude}} \tag{9}$$

- Various decompositions exist in literature: various sets of $\mathcal{A}(\sigma)$ and various $C(\sigma)$

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Colour decompositions

- Standard Model: $SU(3)_C \times SU(2)_W \times U(1)_Y$
- Colour factors: either T^a or f^{abc}

$$[T^a, T^b] = if^{abc} T^c \tag{10}$$

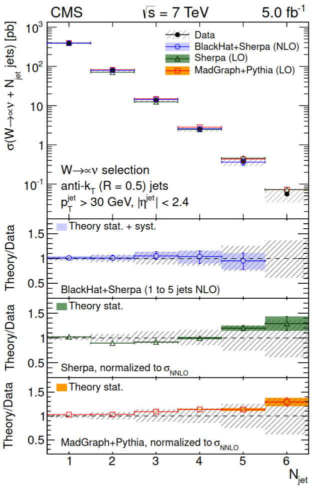
- Decompose amplitude into colour part and kinematical part

$$\mathcal{M} \propto \sum_{\sigma \in S} \underbrace{C(\sigma)}_{\text{colour factor}} \times \underbrace{\mathcal{A}(\sigma)}_{\text{dual amplitude}} \tag{11}$$

$$|\mathcal{M}|^2 \propto \sum_{\sigma_k, \sigma_l} \underbrace{C(\sigma_k, \sigma_l)}_{\text{colour matrix}} \mathcal{A}(\sigma_k)(\mathcal{A}(\sigma_l))^* \tag{12}$$

- Problem: $\sum_{\sigma_k, \sigma_l}$ grows $\sim (n_{\text{ext}}!)^2$

Why do we bother about high-multiplicity?



- Measurements of multi-jet processes (ATLAS and CMS)
- Currently: **exact colour structure** up to 4/5 final state partons + parton showers
- `Comix`: Monte Carlo sampling of colour
- Develop the exact colour treatment
- Avoid factorial growth with n_{ext}:

```
for i=1,...,size(C):
 for j=1,...,size(C):
  M = M + C(i,j)*A(i)*(A(j)*)
```

Why do we bother about high-multiplicity?

- One possible solution: make the colour matrix sparse!
- Expand in N_c^{-2} (large-N_c limit)

$$C(\sigma_k, \sigma_l) = \underbrace{N_c^x}_{\text{Leading colour (LC)}} + \underbrace{N_c^{x-2}}_{\text{Next-to-leading colour (NLC)}} + \mathcal{O}(N_c^{x-4}) \quad \forall k, l$$

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$$C(\sigma_k,\sigma_l) = \begin{pmatrix} \mathrm{LC} & 0 & 0 & 0 & 0 & \mathrm{NLC} \\ 0 & \mathrm{LC} & 0 & \mathrm{NLC} & 0 & 0 \\ 0 & 0 & \mathrm{LC} & 0 & 0 & 0 \\ 0 & \mathrm{NLC} & 0 & \mathrm{LC} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{LC} & 0 \\ \mathrm{NLC} & 0 & 0 & 0 & 0 & \mathrm{NLC} \end{pmatrix}$$

Reduce factorial growth of the colour structure to some milder scaling

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Reduce factorial growth of the colour structure to some milder scaling

- Where do we find the NLC terms, how much simpler does it become and how accurate does it become?

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Reduce factorial growth of the colour structure to some milder scaling

- Where do we find the NLC terms, $\rightarrow$ TODAY
  how much simpler does it become$\rightarrow$ TODAY
  and how accurate does it become? $\rightarrow$ A. Lifson & O. Mattelaer

Colour decompositions

- **Fundamental decomposition** [2].
  → in terms of traces of fundamental matrices T^a
  → non-minimal basis
- **Colour-flow decomposition** [3]
  → same set of dual amplitudes
  → colour factors are simple Kronecker deltas
- **Adjoint or DDM decomposition** [4]
  → for all-gluon amplitudes only
  → minimal basis
- **SU(N_c) multiplets** [5]

[2] M. L. Mangano, S. J. Parke, Z. Xu FERMILAB-PUB-87-052-T
[3] F. Maltoni *et al.* LL-TH-02-7, FERMILAB-PUB-02-197-T
[4] V. Del Duca, L. J. Dixon, F. Maltoni SLAC-PUB-8294, DFTT-53-99
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Objective of project and outline of results

Pinpoint elements in (tree-level) colour matrix of NLC accuracy

- Find rules in fundamental and colour-flow decompositions and compare
- Do this for:
  - all-gluon amplitudes
  - one quark pair plus gluons
  - two quark pairs with distinct flavour, plus gluons
  - two quark pairs with same flavour, plus gluons
- So 2 *4 sets of analyses presented, all based on same procedure:
  1. Matrix-element and conjugate matrix-element
  2. Squared matrix element and colour matrix
  3. Find NLC terms
  4. Count the number of terms at NLC in the colour matrix

Useful identities

- Recall: Fierz identity

$$(T^a)_{ij}(T^a)_{kl} = T_R\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right) \tag{13}$$

set group index $T_R = 1$

Notation

Use $\mathcal{R}, \mathcal{Q}, \mathcal{S}, \mathcal{P}$... to denote strings of fundamental generators

$$\mathcal{R} = T^{a_1} T^{a_2} \dots T^{a_r} \quad , \quad \tilde{\mathcal{R}} = T^{a_r} T^{a_{r-1}} \dots T^{a_1} \quad , \quad \mathrm{len}(\mathcal{R}) = r \tag{14}$$

Useful identities

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- Variations of the Fierz identity:

$$\text{Rule I:} \qquad \text{Tr}[T^a\mathcal{R}]\text{Tr}[T^a\mathcal{S}] = \text{Tr}[\mathcal{R}\mathcal{S}] - \frac{1}{N_c}\text{Tr}[\mathcal{R}]\text{Tr}[\mathcal{S}], \tag{15}$$

$$\text{Rule II:} \qquad \text{Tr}[\mathcal{R}T^a\mathcal{Q}T^a\mathcal{S}] = \text{Tr}[\mathcal{Q}]\text{Tr}[\mathcal{R}\mathcal{S}] - \frac{1}{N_c}\text{Tr}[\mathcal{R}\mathcal{Q}\mathcal{S}], \tag{16}$$

$$\text{Rule IIb:} \qquad \text{Tr}[\mathcal{R}T^aT^a\mathcal{S}] = N_c\text{Tr}[\mathcal{R}\mathcal{S}] + \mathcal{O}(1/N_c) \tag{17}$$

Fundamental decomposition

For n-gluon amplitudes

- Matrix-element

$$\mathcal{M} = g^{n-2} \sum_{\sigma \in S_{n-1}} \mathrm{Tr}[T^{a_1} T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n-1)}}] \mathcal{A}(1, \sigma(1), \ldots, \sigma(n-1)) \qquad (18)$$

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Not a minimal set

Dual amplitudes are related by the Kleiss-Kuijf relation (dual Ward identity)

$$\mathcal{A}(1,2,3,4\dots,n) + \mathcal{A}(2,1,3,4,\dots,n) + \dots + \mathcal{A}(2,3,4,\dots,1,n) = 0 \qquad (19)$$

Fundamental decomposition

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$$\mathcal{A}(1,2,3,4....,n) + \mathcal{A}(2,1,3,4,...,n) + ... + \mathcal{A}(2,3,4,...,1,n) = 0 \qquad (19)$$

- Squared matrix-element

$$|\mathcal{M}|^2 = (g^2)^{n-2} \sum_{k,l=1}^{(n-1)!} C_{kl} \mathcal{A}(1, \sigma_k(1), \ldots, \sigma_k(n-1)) \left(\mathcal{A}(1, \sigma_l(1), \ldots, \sigma_l(n-1))\right)^*$$

- Colour matrix (size $(n-1)! \times (n-1)!$):

$$C_{kl} = \sum_{\text{col.}} \mathrm{Tr}[T^{a_1} T^{a_{\sigma_k(1)}} \ldots T^{a_{\sigma_k(n-1)}}] \left(\mathrm{Tr}[T^{a_1} T^{a_{\sigma_l(1)}} \ldots T^{a_{\sigma_l(n-1)}}]\right)^* \qquad (20)$$

Fundamental decomposition

For *n*-gluon amplitudes

- Colour matrix

$$C_{kl} = \sum_{\text{col.}} \mathrm{Tr}[T^{a_1} T^{a_{\sigma_k(1)}} \dots T^{a_{\sigma_k(n-1)}}] (\mathrm{Tr}[T^{a_1} T^{a_{\sigma_l(1)}} \dots T^{a_{\sigma_l(n-1)}}])^* \tag{21}$$

- LC: $\mathcal{O}(N_c^n)$, NLC: $\mathcal{O}(N_c^{n-2})$
- Leading-colour in all diagonal elements $\sigma_k = \sigma_l$

$$\left(N_c - \frac{1}{N_c}\right)^n + (N_c^2 - 1)\left(\frac{-1}{N_c}\right)^n = N_c^n + \mathcal{O}(N_c^{n-2}) \tag{22}$$

- Diagonal colour factors contain also NLC contribution
- But which other off-diuagonal elements are NLC accurate?

Fundamental decomposition

For n-gluon amplitudes

- NLC: for permutations which are related by a block interchange:[6]

$$\sigma_k \sim \mathcal{R}\mathcal{Q}_1\mathcal{S}\mathcal{Q}_2\mathcal{P} \quad , \quad \sigma_l \sim \mathcal{R}\mathcal{Q}_2\mathcal{S}\mathcal{Q}_1\mathcal{P} \tag{23}$$

with special cases if $\mathcal{S} = \mathbb{1}$

- If $\text{len}(\mathcal{Q}_1) = \text{len}(\mathcal{Q}_2) = 1$: $-N_c^{n-2} + \mathcal{O}(N_c^{n-4})$.
- If $\text{len}(\mathcal{Q}_1) = 1$ *or* $\text{len}(\mathcal{Q}_2) = 1$: $-N_c^{n-2} + \mathcal{O}(N_c^{n-4})$ if $\text{len}(\mathcal{R}) = 1$ and $\text{len}(\mathcal{P}) = 0$, otherwise not NLC
- If $\text{len}(\mathcal{Q}_{1,2}) > 1$: $+N_c^{n-2} + \mathcal{O}(N_c^{n-4})$ if $\text{len}(\mathcal{R}) \neq 1$ and $\text{len}(\mathcal{P}) \neq 0$, otherwise not NLC

[6] A. Labane, aXiv:2008.13640

Results

For n-gluon amplitudes

Look at $0 \rightarrow n$ amplitudes

all-gluon n	Fundamental	Colour-flow
4	6 (6)	6 (6)
5	11 (24)	16 (24)
6	24 (120)	36 (120)
7	50 (720)	71 (720)
8	95 (5040)	127 (5040)
9	166 (40320)	211 (40320)
10	271 (362880)	331 (362880)
11	419 (3628800)	496 (3628800)
12	620 (39916800)	716 (39916800)
13	885 (479001600)	1002 (479001600)
14	1226 (6227020800)	1366 (6227020800)

Colour-flow decomposition

For n-gluon amplitudes

- Same set of dual amplitudes, colour factors are **strings of Kronecker deltas**
- Colour matrix now

$$C(\sigma_k,\sigma_l)=\delta^{i_1}_{j_{\sigma_k(1)}}\delta^{j_{\sigma_k(1)}}_{i_{\sigma_k(2)}}\ldots\delta^{i_{\sigma_k(n-1)}}_{j_1}\left(\delta^{i_1}_{j_{\sigma_l(1)}}\delta^{j_{\sigma_l(1)}}_{i_{\sigma_l(2)}}\ldots\delta^{i_{\sigma_l(n-1)}}_{j_1}\right)^{\dagger}$$

- Elements of order N_c^{n-2} (NLC) are found in same block interchange-related permutations, **but without the exceptions**

Results

For n-gluon amplitudes

Look at $0 \to n$ amplitudes

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4	6 (6)	6 (6)
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Results

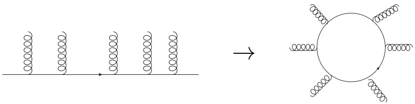
For n-gluon amplitudes

Including the adjoint decomposition: matrix size $(n-2)! \times (n-2)!$

all-gluon			
n	Fundamental	Colour-flow	Adjoint
4	6 (6)	6 (6)	2 (2)
5	11 (24)	16 (24)	5 (6)
6	24 (120)	36 (120)	18 (24)
7	50 (720)	71 (720)	93 (120)
8	95 (5040)	127 (5040)	583 (720)
9	166 (40320)	211 (40320)	4162 (5040)
10	271 (362880)	331 (362880)	31649 (40320)
11	419 (3628800)	496 (3628800)	-
12	620 (39916800)	716 (39916800)	-
13	885 (479001600)	1002 (479001600)	-
14	1226 (6227020800)	1366 (6227020800)	-

Fundamental decompositions

For one quark pair plus *n*-gluon amplitudes

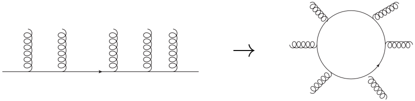


- Matrix-element

$$\mathcal{M}_{1qq} = g^n \sum_{\sigma \in S_n} \left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}\right)_{i_1 j_1} \mathcal{A}_{1qq}(q, \overline{q}, \sigma(1), \ldots, \sigma(n)) \qquad (24)$$

Fundamental decompositions

For one quark pair plus n-gluon amplitudes



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- Squared matrix-element

$$|\mathcal{M}_{1qq}|^2 = (g^2)^n \sum_{k,l=1}^{n!} C_{kl} \mathcal{A}_{1qq}(q, \overline{q}, \sigma_k(1), \dots, \sigma_k(n)) \qquad (25)$$

$$\left(\mathcal{A}_{1qq}(q, \overline{q}, \sigma_l(1), \dots, \sigma_l(n)) \right)^* \qquad (26)$$

- Colour matrix (size $n! \times n!$):

$$C_{kl} = \sum_{\text{col.}} \mathrm{Tr}[T^{a_{\sigma_k(1)}} \dots T^{a_{\sigma_k(n)}} T^{a_{\sigma_l(n)}} \dots T^{a_{\sigma_l(1)}}]. \qquad (27)$$

Fundamental decompositions

For one quark pair plus n-gluon amplitudes

- Colour matrix

$$C_{kl} = \sum_{\text{col.}} \text{Tr}[T^{a_{\sigma_k(1)}} \ldots T^{a_{\sigma_k(n)}} T^{a_{\sigma_l(n)}} \ldots T^{a_{\sigma_l(1)}}]. \tag{28}$$

- LC: $\mathcal{O}(N_c^{n+1})$, NLC: $\mathcal{O}(N_c^{n-1})$
- Leading-colour in all diagonal terms with colour factor

$$N_c \left(N_c - \frac{1}{N_c} \right)^n \tag{29}$$

Fundamental decompositions

For one quark pair plus *n*-gluon amplitudes

- Colour matrix

$$C_{kl} = \sum_{\text{col.}} \mathrm{Tr}[T^{a_{\sigma_k(1)}} \dots T^{a_{\sigma_k(n)}} T^{a_{\sigma_l(n)}} \dots T^{a_{\sigma_l(1)}}]. \tag{28}$$

- LC: $\mathcal{O}(N_c^{n+1})$, NLC: $\mathcal{O}(N_c^{n-1})$
- Leading-colour in all diagonal terms with colour factor

$$N_c \left(N_c - \frac{1}{N_c} \right)^n \tag{29}$$

- NLC terms in block-interchange-related amplitudes

$$\sigma_k \sim \mathcal{R}\mathcal{Q}_1\mathcal{S}\mathcal{Q}_2\mathcal{P} \quad , \quad \sigma_l \sim \mathcal{R}\mathcal{Q}_2\mathcal{S}\mathcal{Q}_1\mathcal{P} \tag{30}$$

with the special cases if $\mathcal{S} = \mathbb{1}$:

- If $\mathrm{len}(\mathcal{Q}_1) = \mathrm{len}(\mathcal{Q}_2) = 1$: $-N_c^{n-1} + \mathcal{O}(N_c^{n-3})$.
- If $\mathrm{len}(\mathcal{Q}_1) = 1$ *or* $\mathrm{len}(\mathcal{Q}_2) = 1$: $\mathcal{O}(N_c^{n-3})$.

Results

For one quark pair plus n-gluon amplitudes

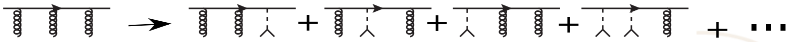
Look at $0 \to q\bar{q} + n$ gluons

$q\bar{q}+ng$		Colour-flow	
n	Fundamental	no external U(1)	one external U(1)
2	2 (2)	4 (5)	3 (5)
3	4 (6)	9 (16)	4 (16)
4	10 (24)	20 (65)	5 (65)
5	24 (120)	41 (326)	6 (326)
6	51 (720)	77 (1957)	7 (1957)
7	97 (5040)	134 (13700)	8 (13700)
8	169 (40320)	219 (109601)	9 (109601)
9	275 (362880)	340 (986410)	10 (986410)
10	424 (3628800)	506 (9864101)	11 (9864101)
11	626 (39916800)	727 (108505112)	12 (108505112)
12	892 (479001600)	1014 (1302061345)	13 (1302061345)

Colour-flow decompositions

For one quark pair plus n-gluon amplitudes

- Quark line → we can have external U(1) gluons!
- Add dual amplitudes with each combination of gluons projected out



with the replacement of SU(N_c) gluons to U(N_c) and U(1) parts

$$\delta^i_j \delta^k_l \rightarrow \delta^i_j \delta^k_l - \frac{1}{N_c} \delta^i_l \delta^k_j \qquad (31)$$

Colour-flow decompositions

For one quark pair plus *n*-gluon amplitudes

- Leading-colour only in diagonal terms with no external U(1) gluons

$$\delta^{i_q}_{j_{\sigma_k(1)}}\delta^{i_{\sigma_k(1)}}_{j_{\sigma_k(2)}}\ldots\delta^{i_{\sigma_k(n-1)}}_{j_{\sigma_k(n)}}\delta^{i_{\sigma_k(n)}}_{j_q}\times\left(\delta^{i_q}_{j_{\sigma_l(1)}}\delta^{i_{\sigma_l(1)}}_{j_{\sigma_l(2)}}\ldots\delta^{i_{\sigma_l(n-1)}}_{j_{\sigma_l(n)}}\delta^{i_{\sigma_l(n)}}_{j_q}\right)^\dagger=N_{\mathbf{c}}^{n+1}, \tag{32}$$

- NLC ($N_{\mathbf{c}}^{n-1}$) in three type:
  - NLC of type 1: $\mathcal{A}(\text{only U}(3))\times\mathcal{A}(\text{only U}(3))^*$
  - NLC of type 2: $\mathcal{A}(\text{only U}(3))\times\mathcal{A}(\text{one U}(1))^*$ (reduces to type 3)
  - NLC of type 3: $\mathcal{A}(\text{one U}(1))\times\mathcal{A}(\text{one U}(1))^*$

Results

For one quark pair plus *n*-gluon amplitudes

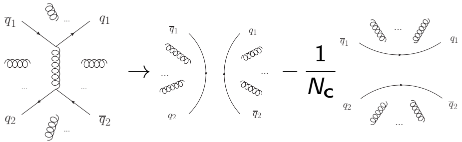
Look at $0 \to q\bar{q} + n$ gluons

$q\bar{q} + ng$		Colour-flow	
n	Fundamental	no external U(1)	one external U(1)
2	2 (2)	4 (5)	3 (5)
3	4 (6)	9 (16)	4 (16)
4	10 (24)	20 (65)	5 (65)
5	24 (120)	41 (326)	6 (326)
6	51 (720)	77 (1957)	7 (1957)
7	97 (5040)	134 (13700)	8 (13700)
8	169 (40320)	219 (109601)	9 (109601)
9	275 (362880)	340 (986410)	10 (986410)
10	424 (3628800)	506 (9864101)	11 (9864101)
11	626 (39916800)	727 (108505112)	12 (108505112)
12	892 (479001600)	1014 (1302061345)	13 (1302061345)

Fundamental decompositions

For two **distinct flavour** quark pairs plus n-gluon amplitudes

- Now we have *two single colour lines* $\to$ internal U(1) gluon
- The internal gluon is decomposed into U(N_c) and U(1) part

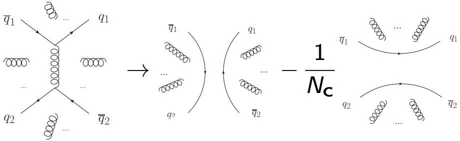


(33)

Fundamental decompositions

For two **distinct flavour** quark pairs plus *n*-gluon amplitudes

- Now we have *two single colour lines* $\rightarrow$ internal U(1) gluon
- The internal gluon is decomposed into U(N_c) and U(1) part



(33)

- The two "quark-ordered" amplitudes

$$\mathcal{M}_{2qq} = \mathcal{M}_1 - \frac{1}{N_c}\mathcal{M}_2 \qquad (34)$$

- Decomposed as

$$\mathcal{M}_1 = \sum_{\sigma \in S_n} \sum_{n_1=0}^{n} c_1(\sigma, n_1)\mathcal{A}_1(\sigma, n_1) \quad , \quad \mathcal{M}_2 = \sum_{\sigma \in S_n} \sum_{n_1=0}^{n} c_2(\sigma, n_1)\mathcal{A}_2(\sigma, n_1)$$

Fundamental decompositions

For two **same flavour** quark pairs plus *n*-gluon amplitudes

- Both a *t*- and *s*-channel contribution

$$\mathcal{M}_{2qq}(\overline{q}q\overline{q}q + ng) = \hat{\mathcal{M}}(\overline{q}_1 q_1 \overline{q}_2 q_2 + ng) - \hat{\mathcal{M}}(\overline{q}_1 q_2 \overline{q}_2 q_1 + ng) \qquad (35)$$

(minus sign from Fermi statistics)

Fundamental decompositions

For two **same flavour** quark pairs plus *n*-gluon amplitudes

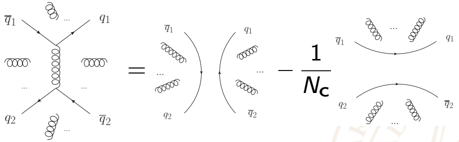
- Both a *t*- and *s*-channel contribution

$$\mathcal{M}_{2qq}(\overline{q}q\overline{q}q + ng) = \hat{\mathcal{M}}(\overline{q}_1 q_1 \overline{q}_2 q_2 + ng) - \hat{\mathcal{M}}(\overline{q}_1 q_2 \overline{q}_2 q_1 + ng) \quad (35)$$

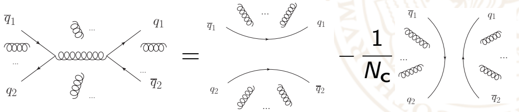
(minus sign from Fermi statistics)

- Decomposed

$$\hat{\mathcal{M}}(\overline{q}_1 q_1 \overline{q}_2 q_2 + ng) =$$



$$\hat{\mathcal{M}}(\overline{q}_1 q_2 \overline{q}_2 q_1 + ng) =$$



(36)

Symmetry factors

- Phase space point generation: identical final state particles need be generated only once!
- $gg \to \underbrace{ggg}_{n_g} + \underbrace{qq}_{n_q} + \underbrace{\overline{qq}}_{n_{\bar{q}}}$
- In fundamental decomposition, simply

$$\frac{N}{n_g!\, n_q!\, n_{\bar{q}}!}, \tag{37}$$

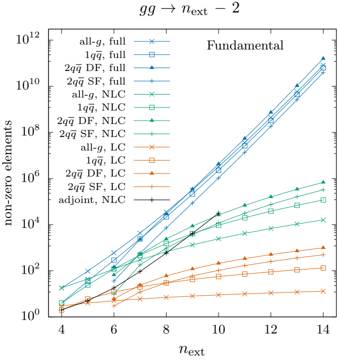
- In colour-flow, case more subtle: consider U(1) gluon amplitudes + order of U(1) gluons not important!

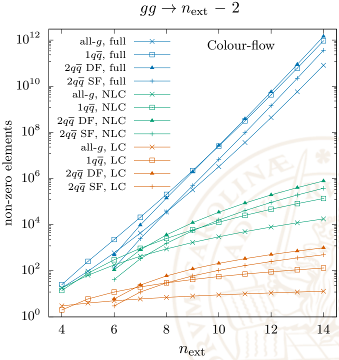
Various QCD-initiated processes

- Include the symmetry factors and **count number of terms in colour matrix at NLC** for the QCD-initiated processes
  - $gg \to n_{\text{ext}} - 2$
  - $qg/\overline{q}g \to n_{\text{ext}} - 2$
  - $q\overline{q} \to n_{\text{ext}} - 2$
  - $qQ/q\overline{Q} \to n_{\text{ext}} - 2$
  - $qq/\overline{qq} \to n_{\text{ext}} - 2$

Results for gg initiated

- For external particles $n_{\text{ext}} \in [4, 14]$
- Blue: full colour   Green: NLC   Red: LC   Black: adjoint NLC

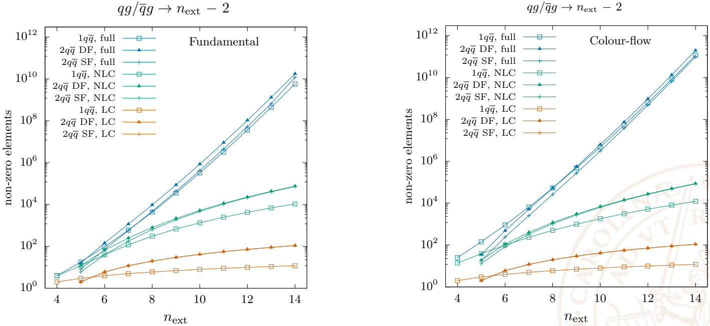




- Turn-over between adjoint and fundamental/colour-flow at $n_{\text{ext}} = 9$

Results for qg initiated

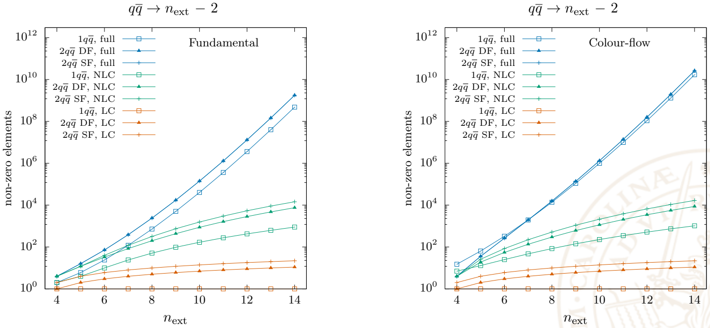
- Blue: full colour   Green: NLC   Red: LC



- Factorial growth for full-colour
- Polynomial scaling with n_{ext} for both LC and NLC ($\sim n_{\text{ext}}^4$)

Results for $q\bar{q}$ initiated

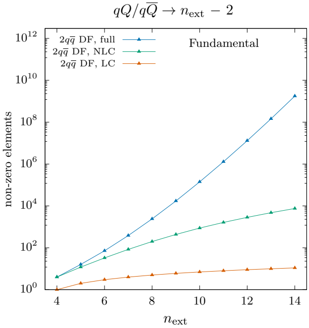
- Blue: full colour Green: NLC Red: LC

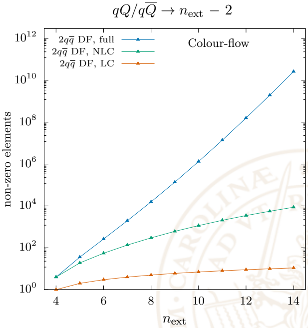


- Colour-flow very slightly less efficient than fundamental decomposition

Results for $qQ/q\overline{Q}$ initiated

- Blue: full colour Green: NLC Red: LC

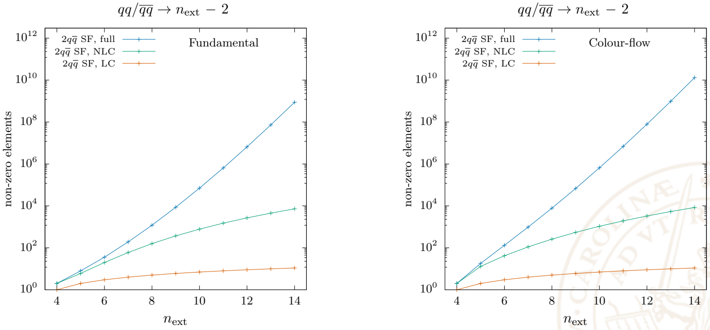




- Already a good efficiency improvement for NLC at $n_{\text{ext}} \sim 6$

Results for $qq/\overline{qq}$ initiated

- Blue: full colour Green: NLC Red: LC



- Same-flavour case has slightly more elements to consider because of symmetrisation of $\mathcal{M}_{1,2}$ amplitudes

Conclusion

- Presented rules to obtain tree-level NLC colour factors for multi-parton processes with up to two quark lines
- Careful treatment of same flavour quark lines needed
- Include phase space symmetrisation to make method more efficient
- Reduce $n!$ complexity of colour sum to $\sim n^4$ at NLC
- Found fundamental decomposition to be (slightly) more efficient than colour-flow and much better than adjoint decomposition
- Sets the path for an efficient event generator for high-multiplicity QCD processes

Outlook

- **Implement this** in MadGraph5_aMC@NLO (together with Andrew Lifson and Olivier Mattelaer)
- Consider **higher orders (NLO)**: colour factors in loops
- Can we find something similar at **NNLC?** (Do we even need NNLC?)
- Sparse colour matrix in **adjoint representation?**
- For more than **2 quark lines?**
- Can we make use of the **Kleiss-Kuijf relation** to further simplify?

Back-up slides

Adjoint decomposition

For n-gluon amplitudes

- The amplitude is now

$$\mathcal{M} = \sum_{\sigma \in S_{n-2}} (F^{a_{\sigma(2)}} \dots F^{a_{\sigma(n-1)}})_{a_1 a_n} \mathcal{A}(1, \sigma(1), \dots, \sigma(n), n), \qquad (38)$$

  with $(F^a)_{bc} = if^{abc}$
- Minimal basis: $(n-2)!$ independent dual amplitudes
- Smaller colour matrix: but **LC not only on diagonal!**
- No found algorithm (yet) to get NLC elements

Colour-flow decompositions

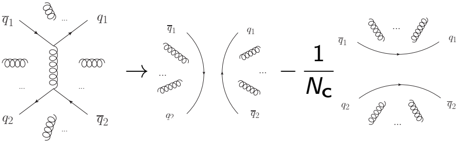
For one quark line plus n-gluon amplitudes: the full projection of U(1) gluons

$$
\begin{aligned}
\mathcal{M}_{1qq} = & \sum_{\sigma\in S_n} \delta^{i_q}_{j_{\sigma(1)}} \delta^{i_{\sigma(1)}}_{j_{\sigma(2)}} \dots \delta^{i_{\sigma(n-1)}}_{j_{\sigma(n)}} \delta^{i_{\sigma(n)}}_{j_q} \mathcal{A}_{1qq}(q,\sigma(1),\dots,\sigma(n),\overline{q}) \\
& + \left(\frac{-1}{N}\right) \sum_{\sigma\in S_n} \delta^{i_q}_{j_{\sigma(1)}} \delta^{i_{\sigma(1)}}_{j_{\sigma(2)}} \dots \delta^{i_{\sigma(n-1)}}_{j_q} \delta^{i_{\sigma(n)}}_{j_{\sigma(n)}} \mathcal{A}_{1qq}(q,\sigma(1),\dots,\sigma(n-1),\overline{q},\sigma(n)) \\
& + \left(\frac{-1}{N}\right)^2 \frac{1}{2!} \sum_{\sigma\in S_n} \delta^{i_q}_{j_{\sigma(1)}} \delta^{i_{\sigma(1)}}_{j_{\sigma(2)}} \dots \delta^{i_{\sigma(n-2)}}_{j_q} \delta^{i_{\sigma(n-1)}}_{j_{\sigma(n-1)}} \delta^{i_{\sigma(n)}}_{j_{\sigma(n)}} \\
& \qquad \mathcal{A}_{1qq}(q,\sigma(1),\dots,\sigma(n-2),\overline{q},\sigma(n-1),\sigma(n)) \\
& + \dots \\
& + \left(\frac{-1}{N}\right)^n \frac{1}{n!} \sum_{\sigma\in S_n} \delta^{i_q}_{j_q} \delta^{i_{\sigma(1)}}_{j_{\sigma(1)}} \dots \delta^{i_{\sigma(n-1)}}_{j_{\sigma(n-1)}} \delta^{i_{\sigma(n)}}_{j_{\sigma(n)}} \mathcal{A}_{1qq}(q,\overline{q},\sigma(1),\dots,\sigma(n)).
\end{aligned}
$$

Fundamental decompositions

For two **distinct flavour** quark pairs plus *n*-gluon amplitudes

- Now we have *two single colour lines* → internal U(1) gluon
- The internal gluon is decomposed into U(N_c) and U(1) part



(39)

- The two "quark-ordered" amplitudes

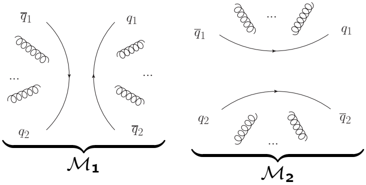
$$\mathcal{M}_{2qq} = \mathcal{M}_1 - \frac{1}{N_c}\mathcal{M}_2 \quad (40)$$

- Decomposed as

$$\mathcal{M}_1 = \sum_{\sigma \in S_n} \sum_{n_1=0}^{n} c_1(\sigma, n_1)\mathcal{A}_1(\sigma, n_1) \quad , \quad \mathcal{M}_2 = \sum_{\sigma \in S_n} \sum_{n_1=0}^{n} c_2(\sigma, n_1)\mathcal{A}_2(\sigma, n_1)$$

Fundamental decompositions

For two **distinct flavour** quark pairs plus *n*-gluon amplitudes



(41)

- The colour factors

$$c_1(\sigma) = \left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n_1)}}\right)_{i_1 j_2} \left(T^{a_{\sigma(n_1+1)}} \ldots T^{a_{\sigma(n)}}\right)_{i_2 j_1} \quad (42)$$

$$c_2(\sigma) = \left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n_1)}}\right)_{i_1 j_1} \left(T^{a_{\sigma(n_1+1)}} \ldots T^{a_{\sigma(n)}}\right)_{i_2 j_2} \quad (43)$$

- The squared amplitude

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2} \sum_{\sigma_k, \sigma_l \in S_{n+1}} \begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) \end{pmatrix} \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^* & -c_1(\sigma_k)c_2(\sigma_l)^*/N_c \\ -c_2(\sigma_k)c_1(\sigma_l)^*/N_c & c_2(\sigma_k)c_2(\sigma_l)^*/N_c^2 \end{pmatrix} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \end{pmatrix} \quad (44)$$

Fundamental decompositions

For two **distinct flavour** quark pairs plus *n*-gluon amplitudes

- Note: **not all diagonal elements the same type now!**
- Leading-colour $\mathcal{O}(N_c^{n+2})$:

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2} \sum_{\sigma_k,\sigma_l \in S_{n+1}}$$

$$\begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) \end{pmatrix} \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^* & -c_1(\sigma_k)c_2(\sigma_l)^*/N_c \\ -c_2(\sigma_k)c_1(\sigma_l)^*/N_c & c_2(\sigma_k)c_2(\sigma_l)^*/N_c^2 \end{pmatrix} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \end{pmatrix}$$

if $\sigma_k = \sigma_l$

- NLC terms $\mathcal{O}(N_c^n)$, investigate block-by-block: appears in each block

Fundamental decompositions

For two **same flavour** quark pairs plus *n*-gluon amplitudes

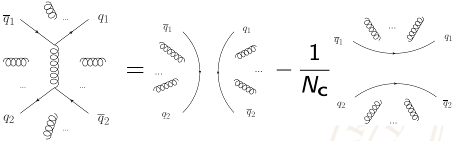
- Both a *t*- and *s*-channel contribution

$$\mathcal{M}_{2qq}(\overline{q}q\overline{q}q+ng)=\hat{\mathcal{M}}(\overline{q}_1q_1\overline{q}_2q_2+ng)-\hat{\mathcal{M}}(\overline{q}_1q_2\overline{q}_2q_1+ng) \qquad (45)$$

(minus sign from Fermi statistics)

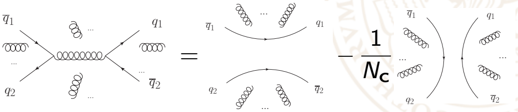
- Decomposed

$$\hat{\mathcal{M}}(\overline{q}_1q_1\overline{q}_2q_2+ng)=$$



(46)

$$\hat{\mathcal{M}}(\overline{q}_1q_2\overline{q}_2q_1+ng)=$$



Fundamental decompositions

For two **same flavour** quark pairs plus *n*-gluon amplitudes

- So then

$$\mathcal{M}_{2qq} = \left(1 + \frac{1}{N_c}\right)(\mathcal{M}_1 - \mathcal{M}_2). \tag{47}$$

- Squared-matrix:

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2}\left(1+\frac{1}{N_c}\right)^2 \sum_{\sigma_k,\sigma_l \in S_{n+1}} \begin{pmatrix}\mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k)\end{pmatrix} \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^* & c_1(\sigma_k)c_2(\sigma_l)^* \\ c_2(\sigma_k)c_1(\sigma_l)^* & c_2(\sigma_k)c_2(\sigma_l)^* \end{pmatrix} \begin{pmatrix}\mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^*\end{pmatrix} \tag{48}$$

- Colour factors include an extra factor $\left(1+\frac{1}{N_c}\right)^2$ here
  $\rightarrow$ LC: $\mathcal{O}(N_c^{n+2})$, non-zero $\mathcal{O}(N_c^{n+1})$

Fundamental decompositions

For two **same flavour** quark pairs plus *n*-gluon amplitudes

- Note: **diagonal elements symmetrized now!**
- Leading-colour $\mathcal{O}(N_c^{n+2})$:

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2}\left(1+\frac{1}{N_c}\right)^2 \sum_{\sigma_k,\sigma_l\in S_{n+1}} \begin{pmatrix}\mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k)\end{pmatrix} \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^* & c_1(\sigma_k)c_2(\sigma_l)^* \\ c_2(\sigma_k)c_1(\sigma_l)^* & c_2(\sigma_k)c_2(\sigma_l)^* \end{pmatrix} \begin{pmatrix}\mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^*\end{pmatrix} \quad (49)$$

  if $\sigma_k = \sigma_l$
- NLC terms $\mathcal{O}(N_c^{n+1}) + \mathcal{O}(N_c^n)$, investigate block-by-block: appears in every block

Colour decompositions

For two **distinct flavour** quark pairs plus n-gluon amplitudes

- Same set of dual amplitudes as for fundamental decomposition

$$\mathcal{M}_{2qq} = \mathcal{M}_1 - \frac{1}{N_c}\mathcal{M}_2 \tag{50}$$

- Once again, external gluons are projected out

$$\mathcal{M}_1 \to \mathcal{M}_1 - \frac{1}{N_c} \sum_{\bar{\sigma}\in\bar{S}_{n+1}} c_1^1(\bar{\sigma})\mathcal{A}_1^1(\bar{\sigma}), \tag{51}$$

- For NLC, it turns out that a single U(1) projection is enough
- Colour factor for this dual amplitude

$$c_1^1(\bar{\sigma}) \quad = \delta^{i_{q_1}}_{j_{\sigma(1)}} \ldots \delta^{i_{\sigma(n)}}_{j_{q_1}} \delta^{i_{\sigma(n+1)}}_{j_{\sigma(n+1)}}, \tag{52}$$

with colourless external U(1) indices

Colour decompositions

For two **distinct flavour** quark line plus n-gluon amplitudes

- Matrix element

$$\mathcal{M}_{2qq} = \sum_{\sigma \in S_{n+1}} c_1(\sigma)\mathcal{A}_1(\sigma) - \frac{1}{N_c}\sum_{\sigma \in S_{n+1}} c_2(\sigma)\mathcal{A}_2(\sigma) - \frac{1}{N_c}\sum_{\bar{\sigma} \in \bar{S}_{n+1}} c_1^1(\bar{\sigma})\mathcal{A}_1^1(\bar{\sigma}),$$

- Squared matrix-element

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n-2}\sum_{\sigma_k,\sigma_l} \begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) & \mathcal{A}_1^1(\bar{\sigma}_k) \end{pmatrix}$$

$$\begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^\dagger & -c_1(\sigma_k)c_2(\sigma_l)^\dagger/N_c & -c_1(\sigma_k)c_1^1(\bar{\sigma}_l)^\dagger/N_c \\ -c_2(\sigma_k)c_1(\sigma_l)^\dagger/N_c & c_2(\sigma_k)c_2(\sigma_l)^\dagger/N_c^2 & c_2(\sigma_k)c_1^1(\bar{\sigma}_l)^\dagger/N_c^2 \\ -c_1^1(\bar{\sigma}_k)c_1(\sigma_l)^\dagger/N_c & c_1^1(\bar{\sigma}_k)c_2(\sigma_l)^\dagger/N_c^2 & c_1^1(\bar{\sigma}_k)c_1^1(\bar{\sigma}_l)^\dagger/N_c^2 \end{pmatrix} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \\ \mathcal{A}_1^1(\bar{\sigma}_l)^* \end{pmatrix},$$

- Leading-colour (N_c^{n+2}) for $\sigma_k = \sigma_l$
- NLC (N_c^n) needs a careful analysis block-by-block

Colour decompositions

For two **same flavour** quark line plus n-gluon amplitudes

- Very similar to the distinct flavour case, but we also need to U(1) project the $\mathcal{M}_2$ amplitude

$$\mathcal{M}_2 \to \mathcal{M}_2 - \frac{1}{N_\mathrm{c}} \sum_{\bar{\sigma} \in \bar{S}_{n+\mathbf{1}}} c_2^1(\bar{\sigma}) \mathcal{A}_2^1(\bar{\sigma}), \tag{53}$$

- 2: Squared matrix-element

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n-2} \left(1 + \frac{1}{N_\mathrm{c}}\right)^2 \sum_{\sigma_k, \sigma_l} \begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) & \mathcal{A}_1^1(\bar{\sigma}_k) & \mathcal{A}_2^1(\bar{\sigma}_k) \end{pmatrix} \mathbb{C} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \\ \mathcal{A}_1^1(\bar{\sigma}_l)^* \\ \mathcal{A}_2^1(\bar{\sigma}_l)^* \end{pmatrix} \tag{54}$$

(Again, colourr factors no longer monomials in N_c)

Colour decompositions

For two **same flavour** quark line plus n-gluon amplitudes

- Colour matrix

$$\mathbb{C} = \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^\dagger & -c_1(\sigma_k)c_2(\sigma_l)^\dagger & -c_1(\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_C & c_1(\sigma_k)c_2^1(\bar\sigma_l)^\dagger/N_C \\ -c_2(\sigma_k)c_1(\sigma_l)^\dagger & c_2(\sigma_k)c_2(\sigma_l)^\dagger & -c_2(\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_C & -c_2(\sigma_k)c_2^1(\bar\sigma_l)^\dagger/N_C \\ -c_1^1(\bar\sigma_k)c_1(\sigma_l)^\dagger/N_C & c_1^1(\bar\sigma_k)c_2(\sigma_l)^\dagger/N_C & c_1^1(\bar\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_C^2 & -c_1^1(\bar\sigma_k)c_2^1(\bar\sigma_l)^\dagger/N_C^2 \\ c_2^1(\bar\sigma_k)c_1(\sigma_l)^\dagger/N_C & -c_2^1(\bar\sigma_k)c_2(\sigma_l)^\dagger/N_C & -c_2^1(\bar\sigma_k)c_1^1(\bar\sigma_l)^\dagger/N_C^2 & c_2^1(\bar\sigma_k)c_2^1(\bar\sigma_l)^\dagger/N_C^2 \end{pmatrix}. \quad (55)$$

- Leading-colour: $\mathcal{O}(N_C^{n+1})$ on first two block diagonal elements
- NLC $\mathcal{O}(N_C^n)$ is examined block-by-block

Non-zero elements without phase-space symmetrisation

$q\bar{q}\,Q\overline{Q}+ng$											Fundamental:		$\mathcal{A}_1 \mid \mathcal{A}_2$ types
	$\min(n_1, n-n_1)$												
n	0 ($\mathcal{A}_1$)	0 ($\mathcal{A}_2$)	1 ($\mathcal{A}_1$)	1 ($\mathcal{A}_2$)	2 ($\mathcal{A}_1$)	2 ($\mathcal{A}_2$)	3 ($\mathcal{A}_1$)	3 ($\mathcal{A}_2$)	4 ($\mathcal{A}_1$)	4 ($\mathcal{A}_2$)	5 ($\mathcal{A}_1$)	5 ($\mathcal{A}_2$)	
0	2	2											(2)
1	3	3											(4)
2	7	4	6	5									(12)
3	15	5	15	7									(48)
4	31	6	32	9	33	10							(240)
5	60	7	62	11	64	13							(1440)
6	108	8	111	13	114	16	115	17					(10080)
7	182	9	186	15	190	19	192	21					(80640)
8	290	10	295	17	300	22	303	25	304	26			(725760)
9	441	11	447	19	453	25	457	29	459	31			(7257600)
10	645	12	652	21	659	28	664	33	667	36	668	37	(79833600)

Table: Number of non-zero elements in a single row of the colour matrix for $q\bar{q}\,Q\overline{Q}+ng$ (distinct flavours) up to NLC accuracy, $\mathcal{O}(N_c^n)$ in the fundamental representation.

Non-zero elements without phase-space symmetrisation

$q\bar{q}Q\overline{Q}+ng$	Colour-flow: $\mathcal{A}_1$, $\mathcal{A}_1^1$ | $\mathcal{A}_2$ types											
n	$\min(n_1, n-n_1)$ 0		1		2		3		4		5	
0	2 , -	2										(2)
1	5 , 3	3										(6)
2	11 , 4	4	12 , -	5								(22)
3	23 , 5	5	25 , 5	7								(98)
4	45 , 6	6	48 , 6	9	49 , -	10						(522)
5	82 , 7	7	86 , 7	11	88 , 7	13						(3262)
6	140 , 8	8	145 , 8	13	148 , 8	16	149 , -	17				(23486)
7	226 , 9	9	232 , 9	15	236 , 9	19	238 , 9	21				(191802)
8	348 , 10	10	355 , 10	17	360 , 10	22	363 , 10	25	364 , -	25		(1753618)
9	515 , 11	11	523 , 11	19	529 , 11	25	533 , 11	29	535 , 11	31		(17755382)
10	737 , 12	12	746 , 12	21	753 , 12	28	758 , 12	33	761 , 12	36	762 , - | 37	(197282022)

Table: Number of non-zero elements in a single row of the colour matrix for $q\bar{q}Q\overline{Q}+ng$ (distinct flavours) up to NLC accuracy in the colour-flow representation