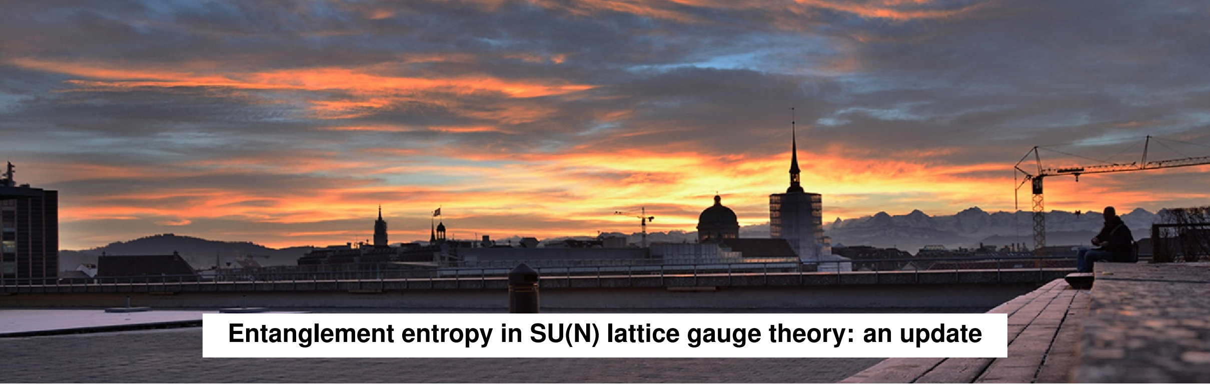

Entanglement entropy in SU(N) lattice gauge theory: an update

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Tobias Rindlisbacher[1], Niko Jokela[2], Kari Rummukainen[2], and Ahmed Salami[2]

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Nordic Lattice Meeting 2024, June 10-12, 2024

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UNIVERSITY OF HELSINKI

What is entanglement?

➔ Quantum physical implementation of conservation laws

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- Decay of spin-0 particle: $s = 0$

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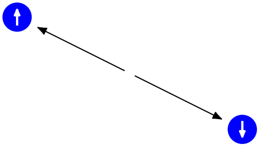

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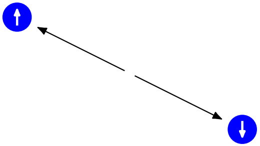

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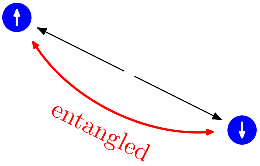

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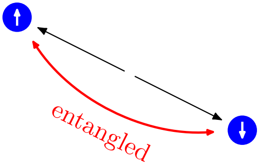

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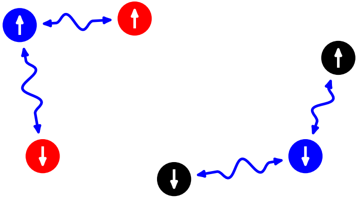

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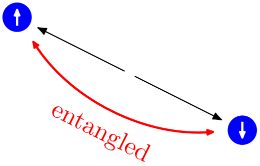

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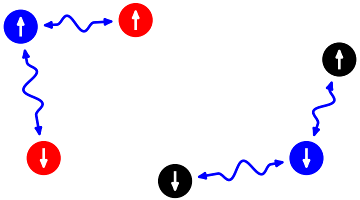
➔ correlations



What is entanglement entropy?

- Preliminaries:

  Hilbert space: $\mathcal{H}$, state vector: $|\psi\rangle \in \mathcal{H}$

  Density matrix:

  $\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \quad , \quad |\psi_i\rangle \in \mathcal{H} \quad \forall\, i \quad , \quad \sum_i p_i = 1$

  $\mathrm{tr}(\rho) = 1$

  pure state: $\rho = |\psi\rangle\langle\psi|$

  ➔ $\rho^2 = \rho$ (projector) ➔ $\mathrm{tr}\big(\rho^2\big) = 1$

  mixed state: $\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$

  ➔ $\rho^2 \neq \rho$ (not projector) ➔ $\mathrm{tr}\big(\rho^2\big) < 1$

Introduction

What is entanglement entropy?

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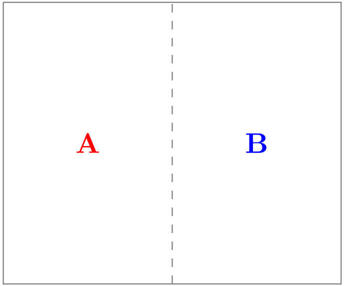
  pick pure state: $|\psi\rangle_{AB} \in \mathcal{H}_{AB}$

  pick orthonormal bases: $|n\rangle_A \in \mathcal{H}_A$, $|m\rangle_B \in \mathcal{H}_B$

  ➔ $|\psi\rangle_{AB} = \sum_{mn} a_{mn} \, |m\rangle_A \otimes |n\rangle_B \quad , \quad \sum_{mn} |a_{mn}|^2 = 1$

  ➔ $\rho_{AB} = |\psi\rangle_{AB}\langle\psi| = \sum_{mnkl} a_{mn} a^*_{kl} |m\rangle_A\langle k| \otimes |n\rangle_B\langle l|$

  (notation: $|\psi\rangle_C\langle\psi| = |\psi\rangle_C \otimes {}_C\langle\psi|$)



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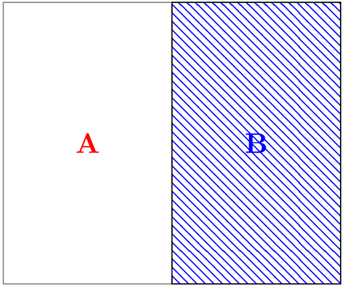

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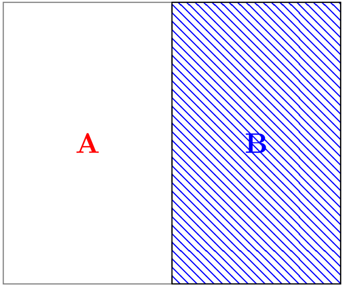

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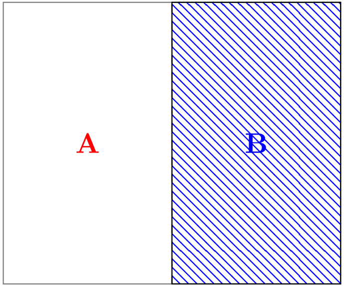

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➔ Purity: $\mathrm{tr}\left(\rho_A^2\right)$

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➔ Entanglement entropy: $S_{EE}(A) = -\lim_{s\to 1} \frac{\partial \log \mathrm{tr}(\rho_A^s)}{\partial s} = \lim_{s\to 1} \frac{\partial((s-1)H_s(A))}{\partial s} = \lim_{s\to 1} H_s(A)$

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Entanglement entropy on the lattice

Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

- SU(N) gauge theory on $N_s^{d-1} \times N_t$ lattice

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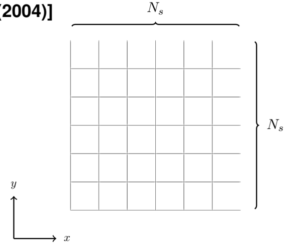

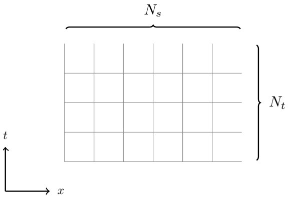

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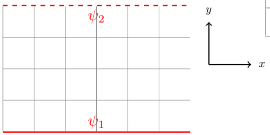
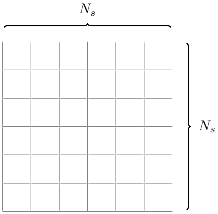
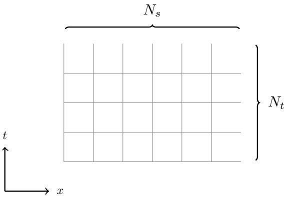
  ➔ Density matrix element:

$$\langle \psi_1 | \rho | \psi_2 \rangle = \int_{\substack{U(\bar{x}, N_t) = \psi_2(\bar{x}) \\ U(\bar{x},\ 0\ ) = \psi_1(\bar{x})}} \mathcal{D}[U]\ \mathrm{e}^{-S_G[U]} =$$







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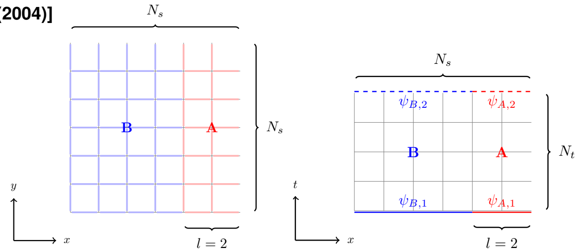

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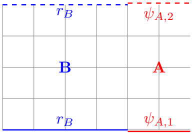

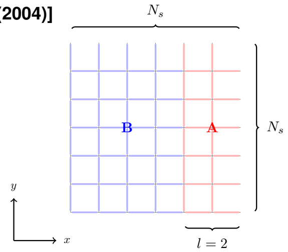

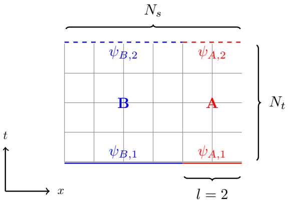

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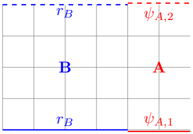

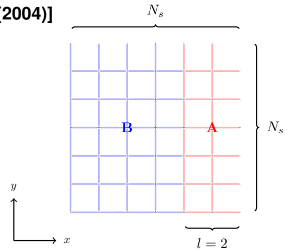

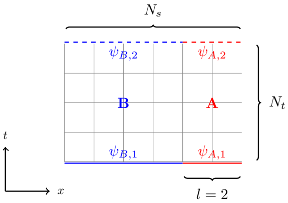

  ➔ Entanglement entropy:

$$S_{EE} = -\,\mathrm{tr}_A(\rho_A \log \rho_A) \quad \text{(how ?)}$$

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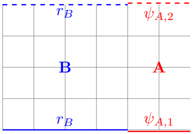


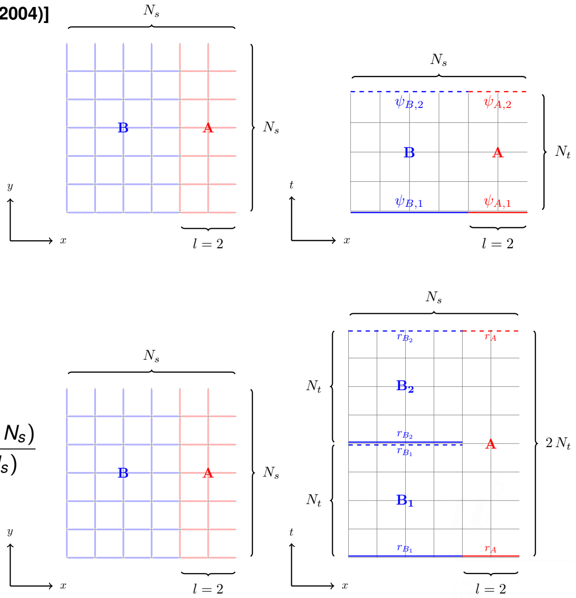
  ➔ Replica method for s-th Rényi entropy:

$$H_s(l, N_t, N_s) = \frac{1}{1-s} \log \mathrm{tr}(\rho_A^s) = \frac{1}{1-s} \log \frac{Z_c(l, s, N_t, N_s)}{Z^s(N_t, N_s)}$$

with "cut partition function" $Z_c(l, s, N_t, N_s)$

$\rightarrow \quad Z_c(l = 0, s, N_t, N_s) = Z^s(N_t, N_s) \quad \forall s \in \mathbb{N}$

$\rightarrow \quad Z_c(l = N_s, s, N_t, N_s) = Z(s\, N_t, N_s) \quad \forall s \in \mathbb{N}$



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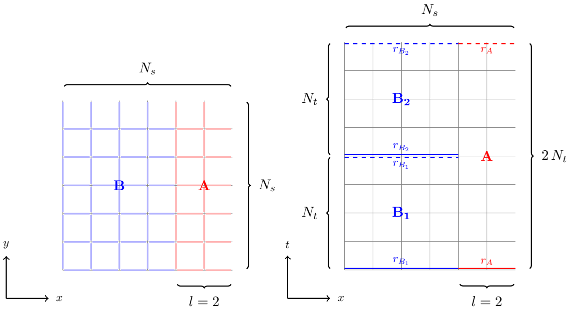
➔ measure free energy difference



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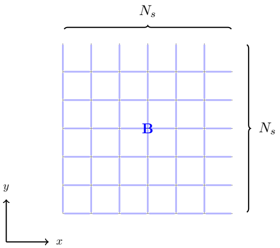

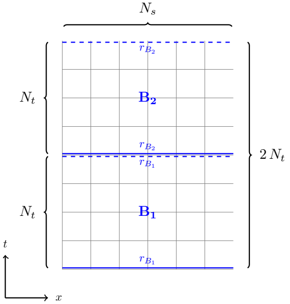

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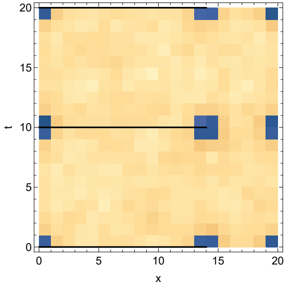
Issue: UV-divergent piece $\frac{S_{EE}}{|\partial A|} = \frac{C_0}{\epsilon^2} - \frac{C}{l^q} + \text{(finite)}$



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Issue: UV-divergent piece $\frac{S_{EE}}{|\partial A|} = \frac{C_0}{\epsilon^2} - \frac{C}{l^q} + \text{(finite)}$

➔ Instead of EE, measure discrete derivative w.r.t. $l > 0$:

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Entanglement entropy on the lattice

Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

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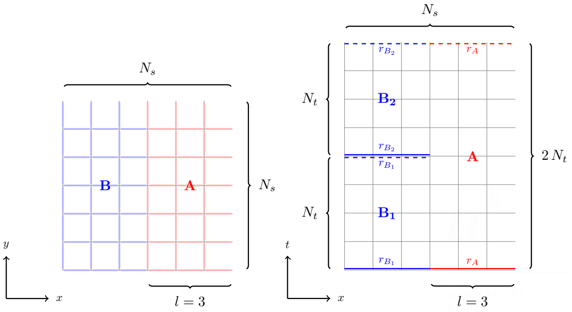

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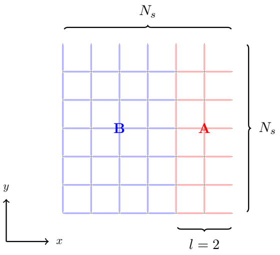

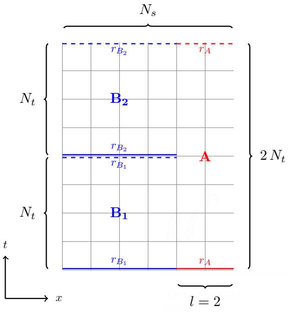

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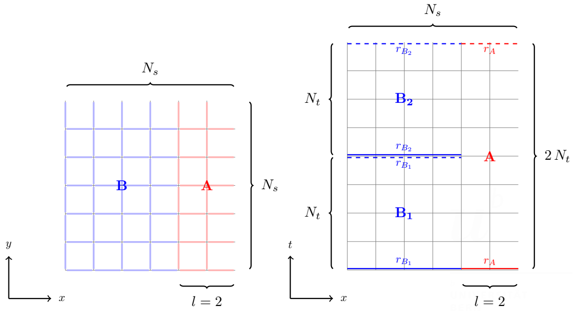

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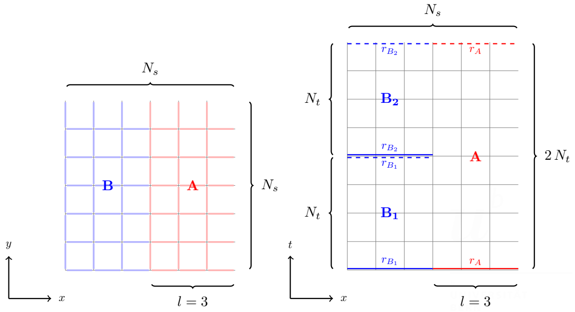

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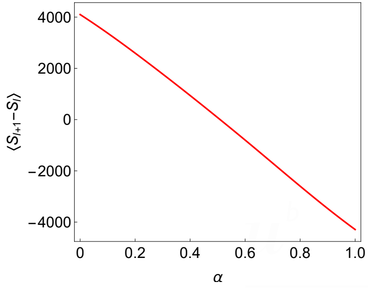

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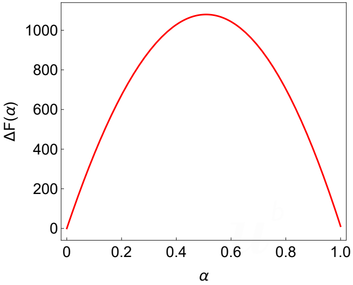

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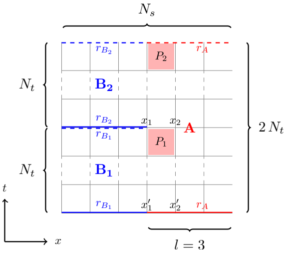
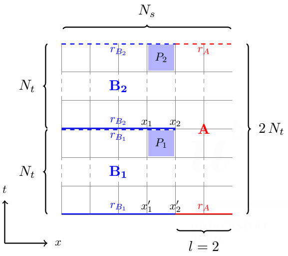
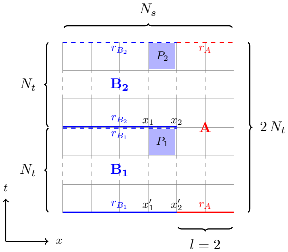
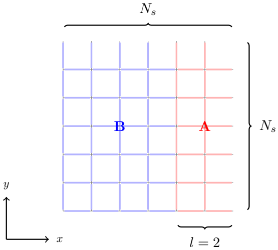




Entangling surface deformation

How can we avoid (huge) free energy barriers?





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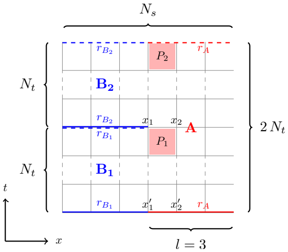

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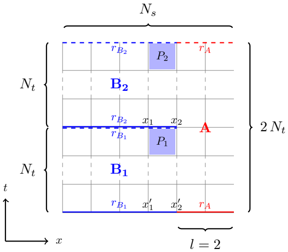

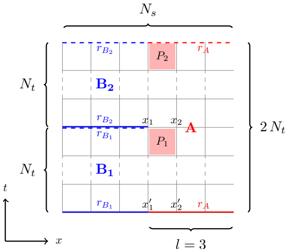

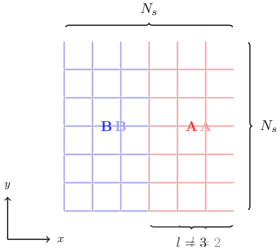

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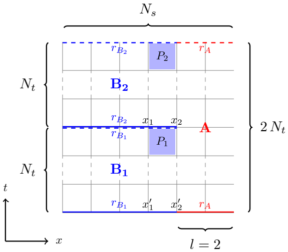

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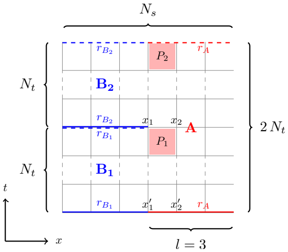

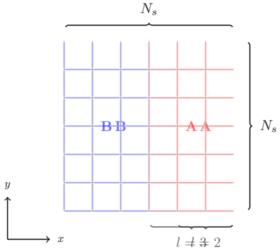

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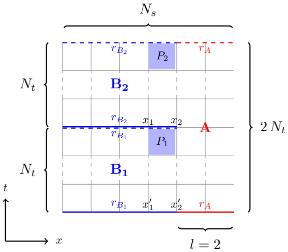

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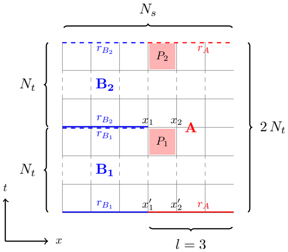

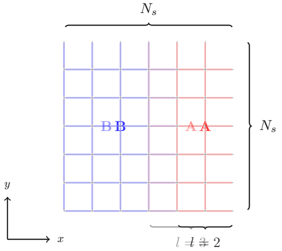

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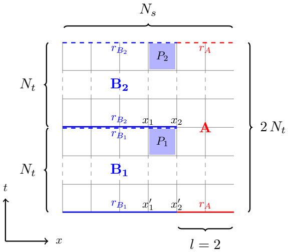

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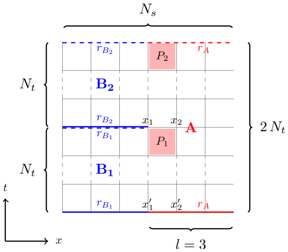

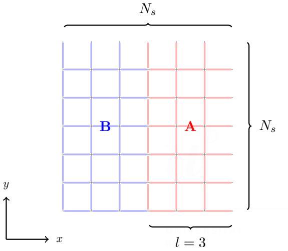

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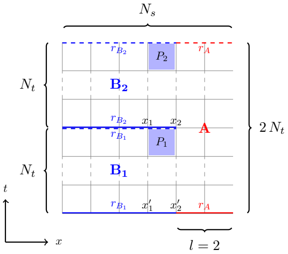

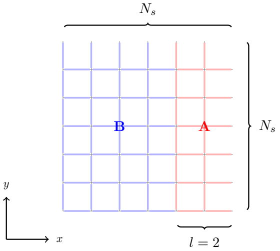

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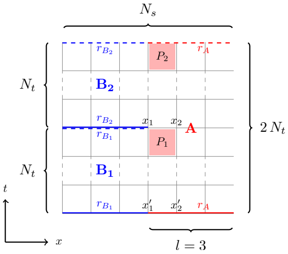

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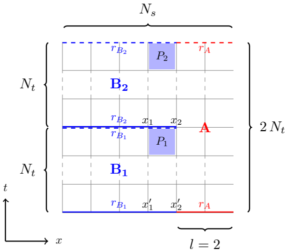

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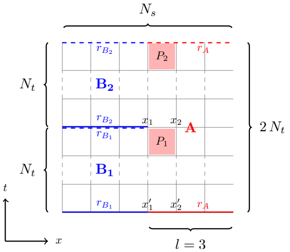

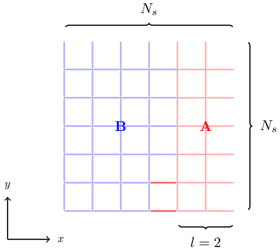

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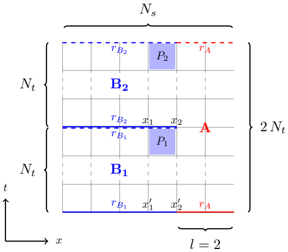

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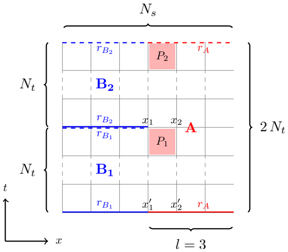

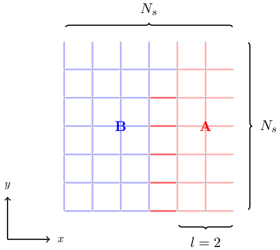

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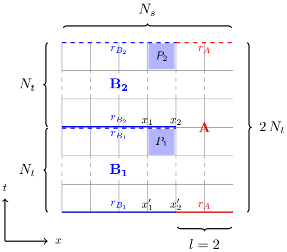

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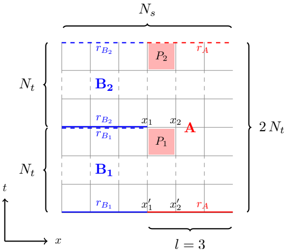

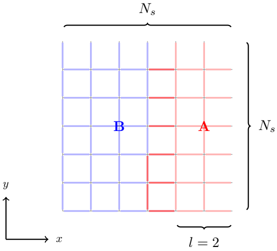

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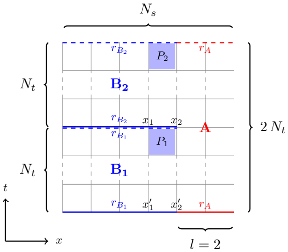

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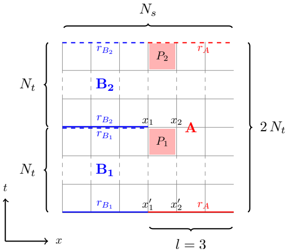

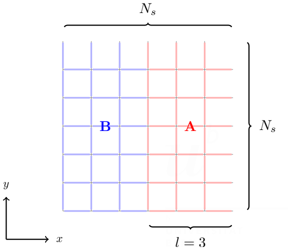

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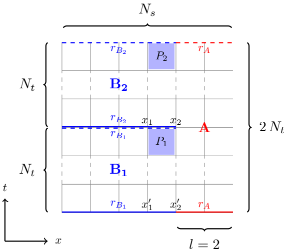

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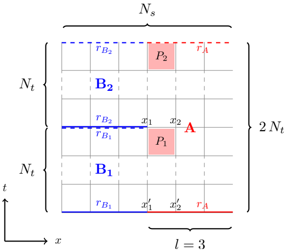
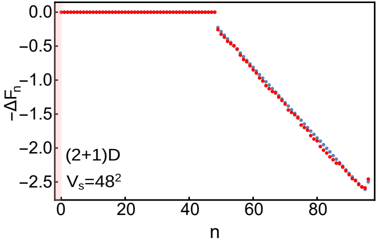
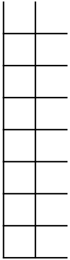
➔ Examples for specific ordering:

➔ in (2+1) dimensions

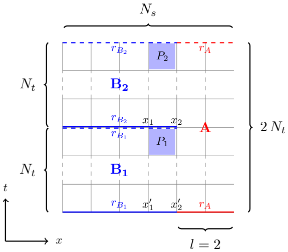






Entangling surface deformation

How can we avoid (huge) free energy barriers?



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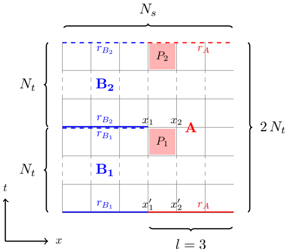

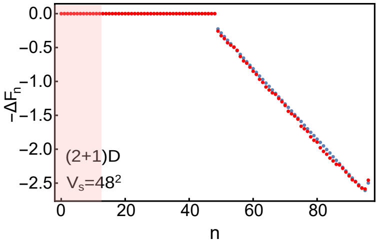

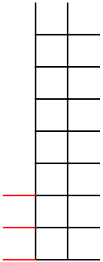

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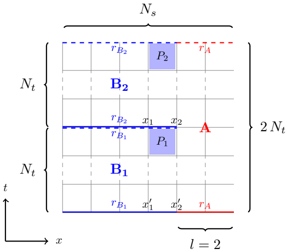

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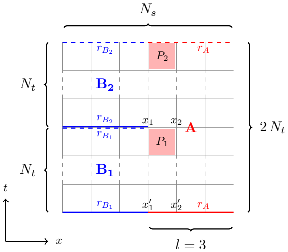

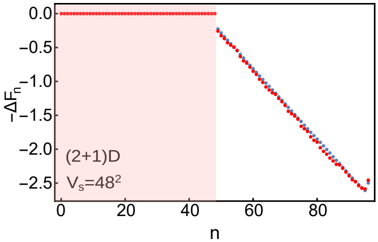

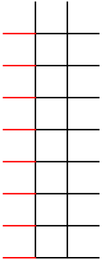

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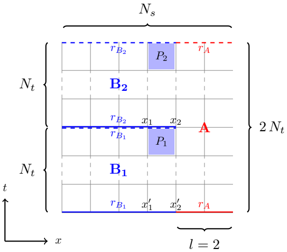

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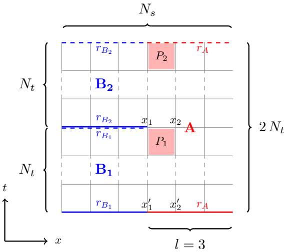

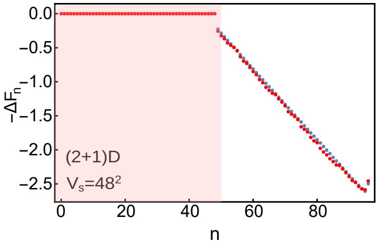

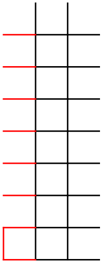

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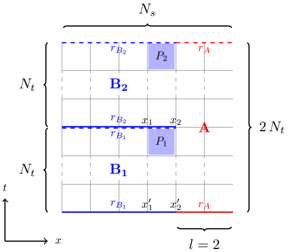

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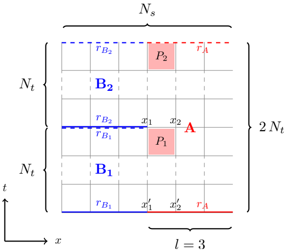

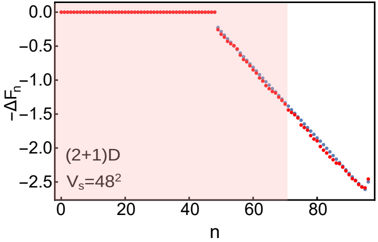

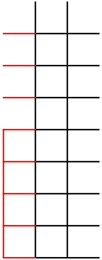

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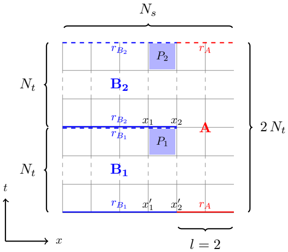

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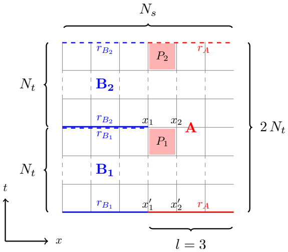

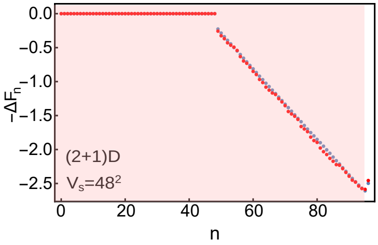

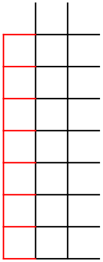

Entangling surface deformation

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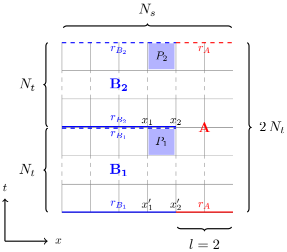

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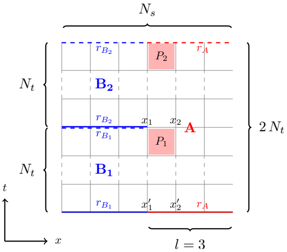

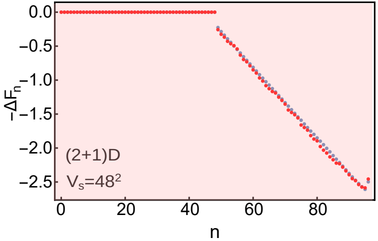

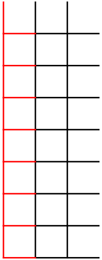

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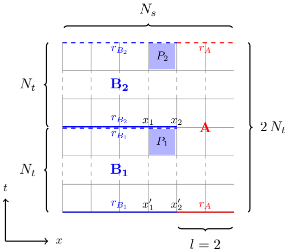

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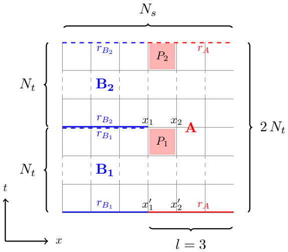

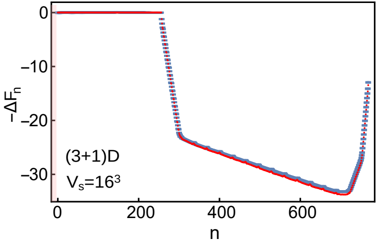

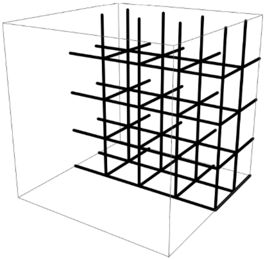

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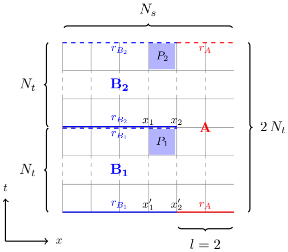

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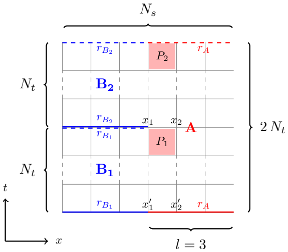

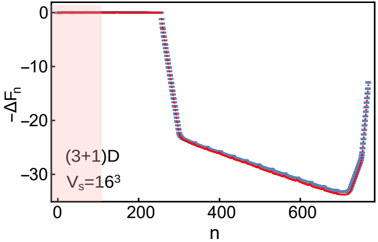

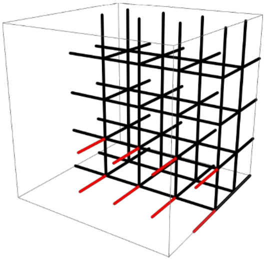

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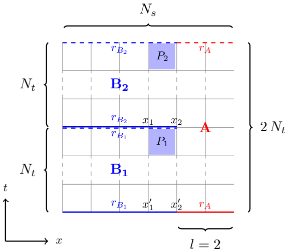

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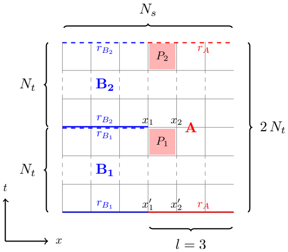

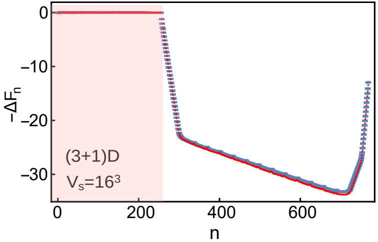

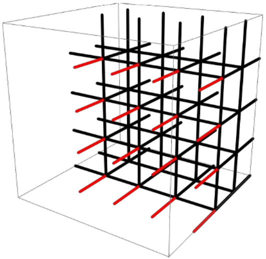

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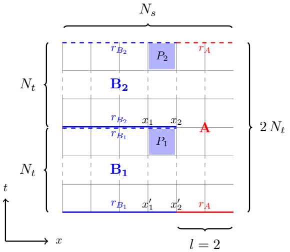

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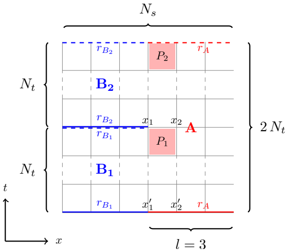

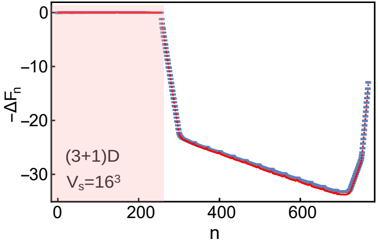

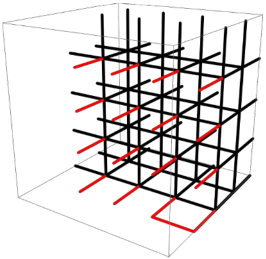

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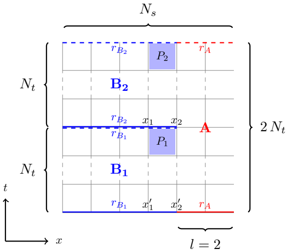

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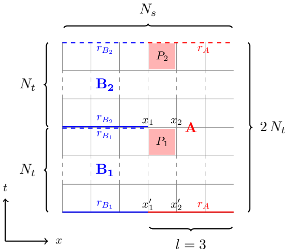

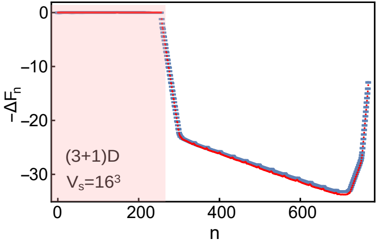

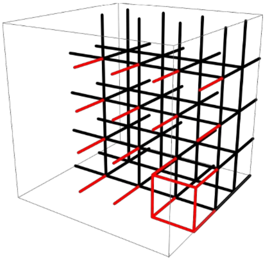

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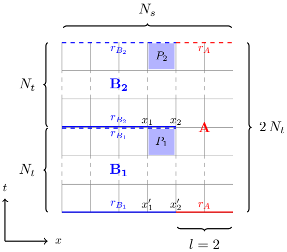

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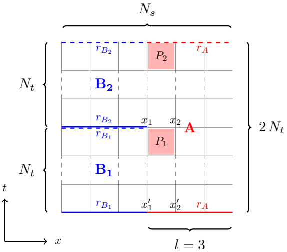

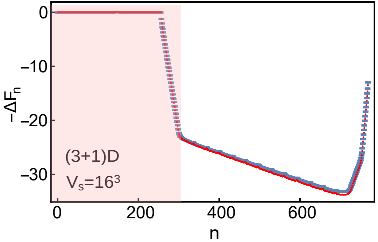

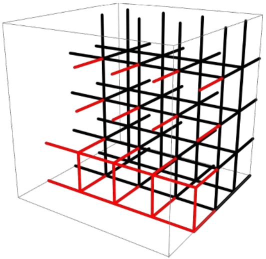

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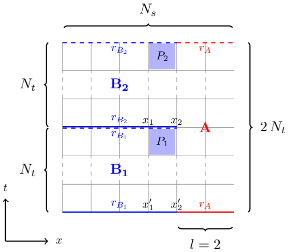

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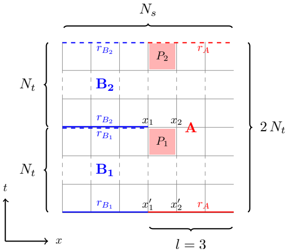

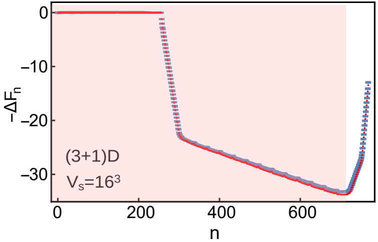

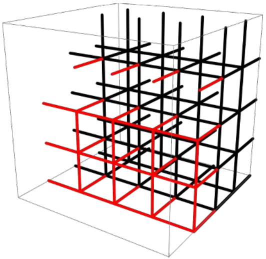

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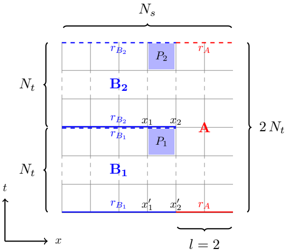

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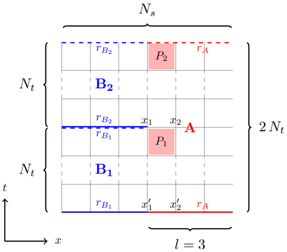

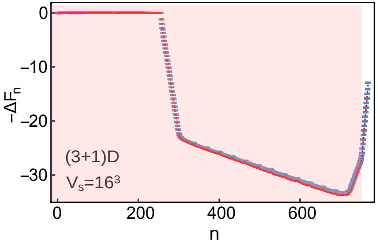

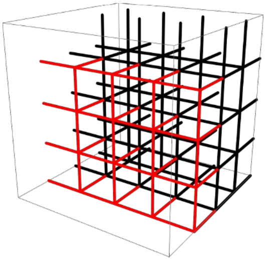

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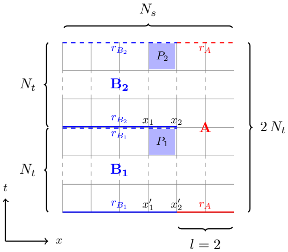

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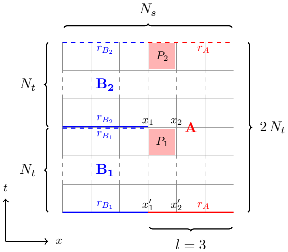
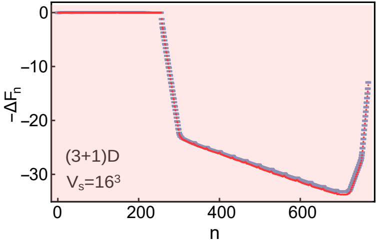
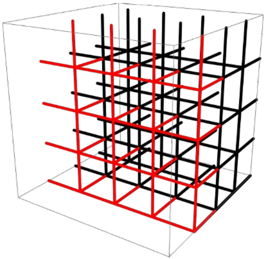
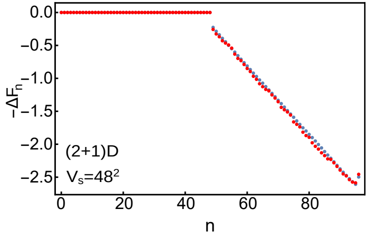


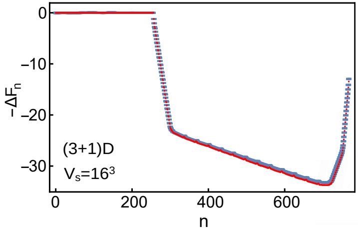




Free-energy plateau

- Why does the free energy initially not change?





Entangling surface deformation

Free-energy plateau

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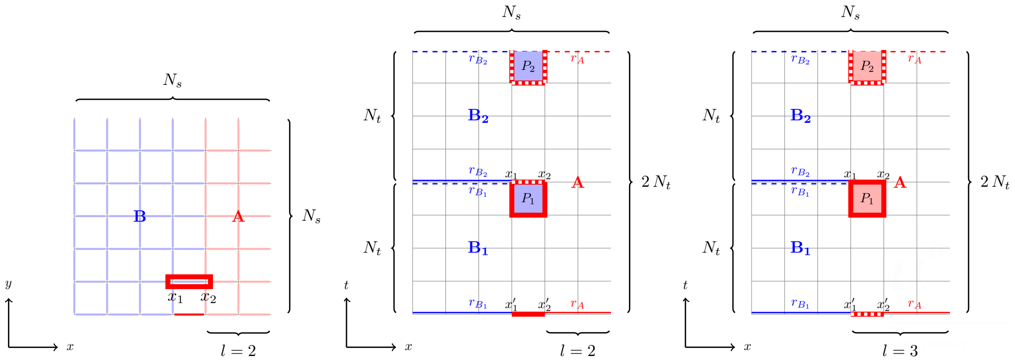
  Change of temp. BC over spatial link $(x_1 \to x_2) \Leftrightarrow P_1, P_2$ swap their upper links.



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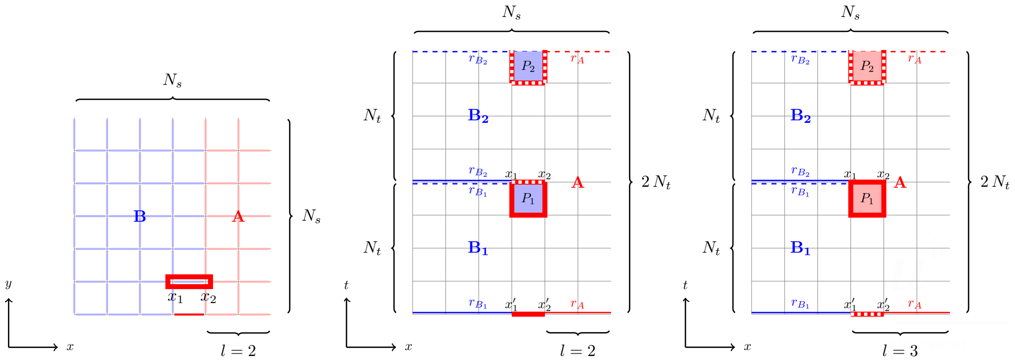

Entangling surface deformation

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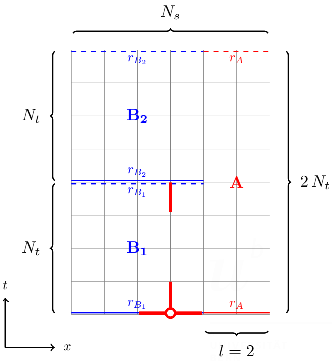

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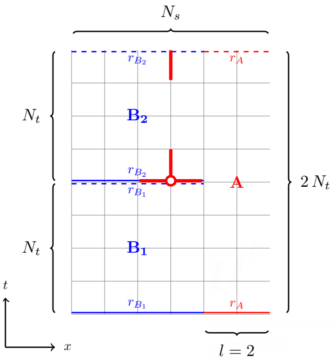

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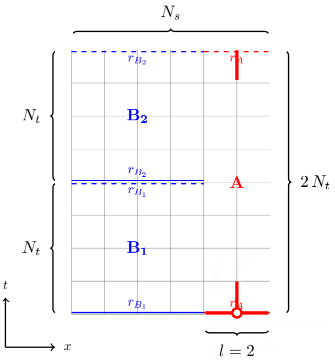

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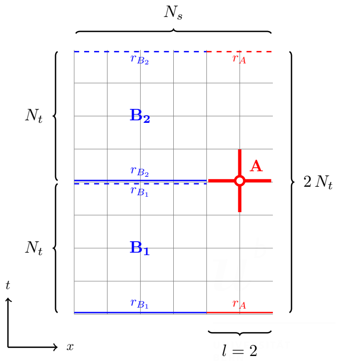

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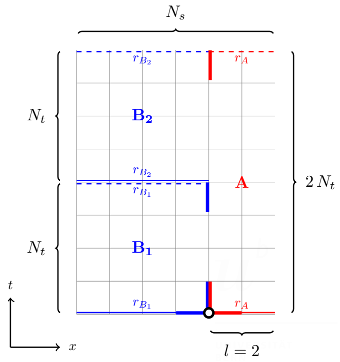

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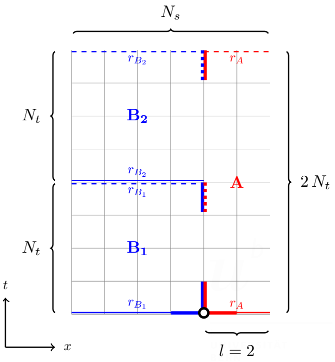

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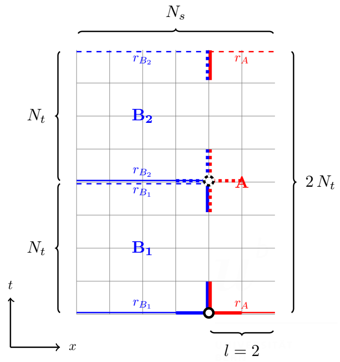

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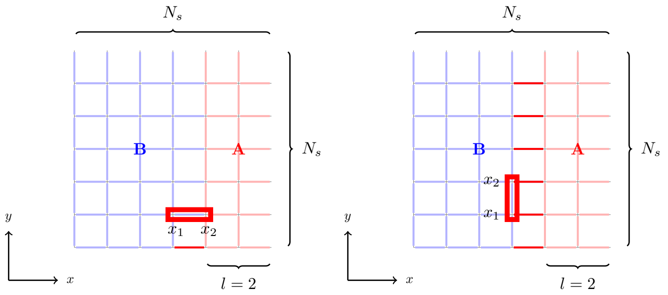

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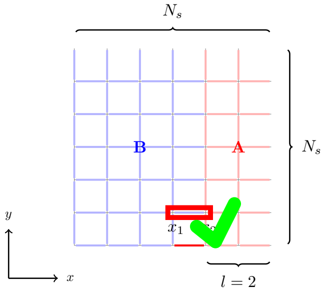
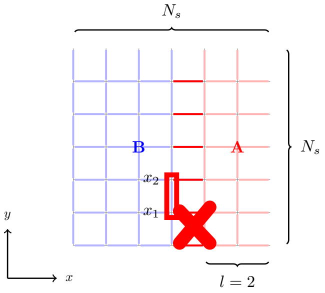
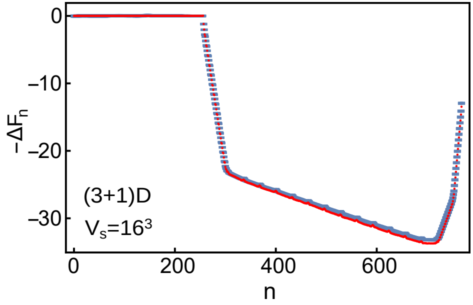




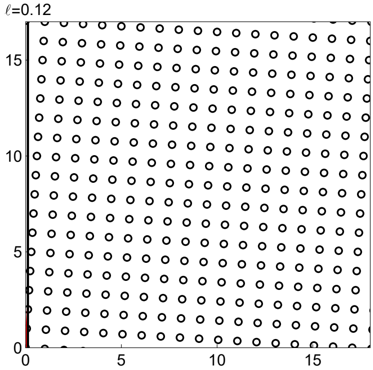
Avoiding remnant free energy barriers



Avoiding remnant free energy barriers

- Tilt lattice with respect to principal directions of "torus"

  ➔ example for (2+1)d lattice:



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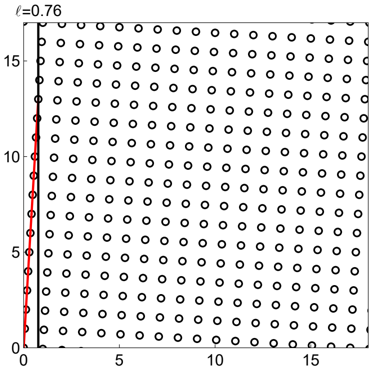

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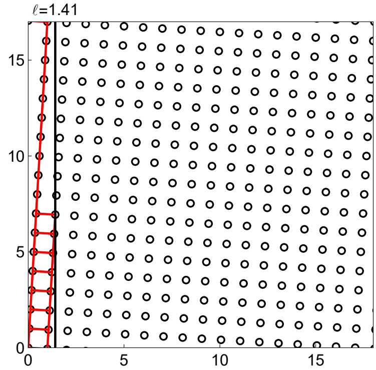

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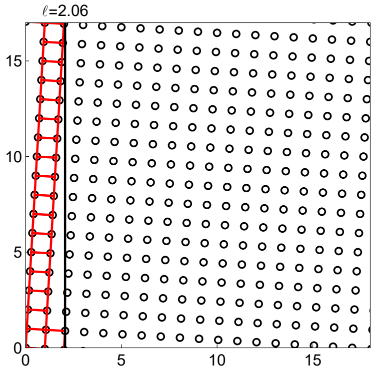

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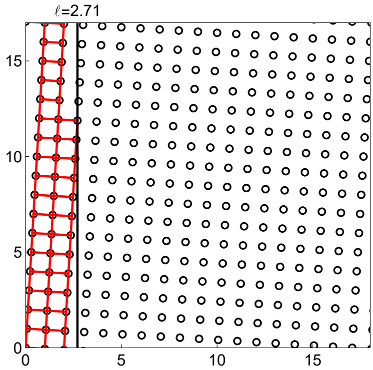

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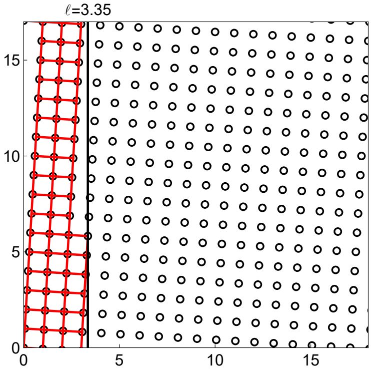

Entangling surface deformation

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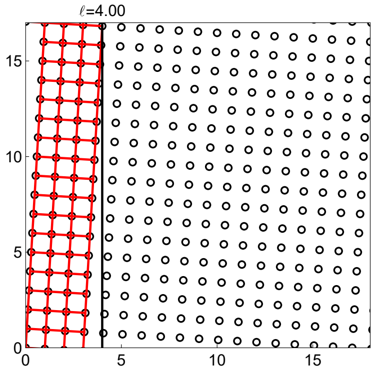

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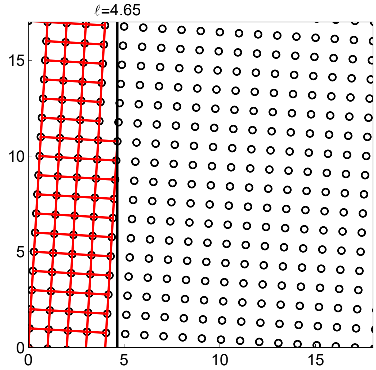

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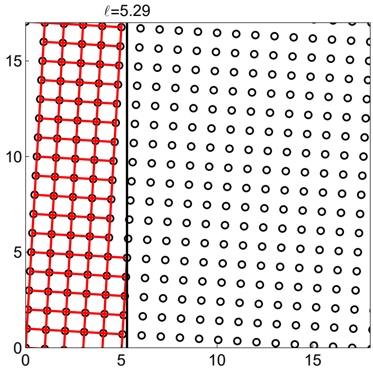

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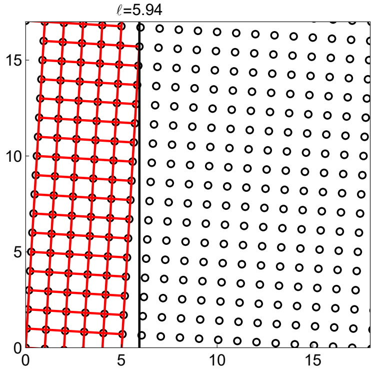

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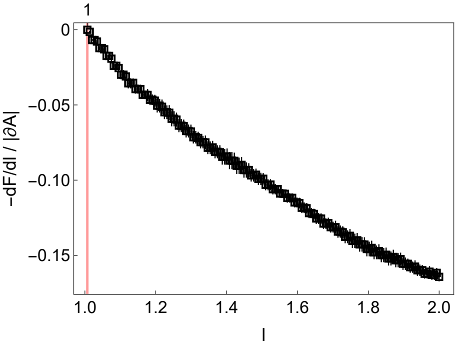

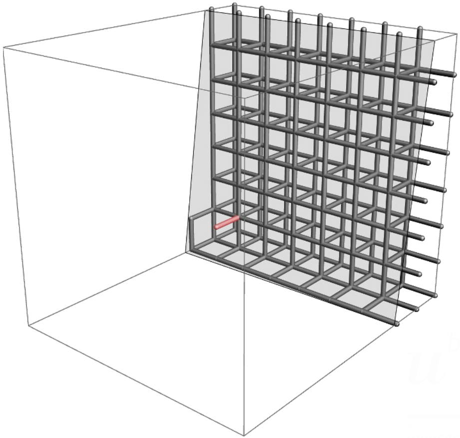

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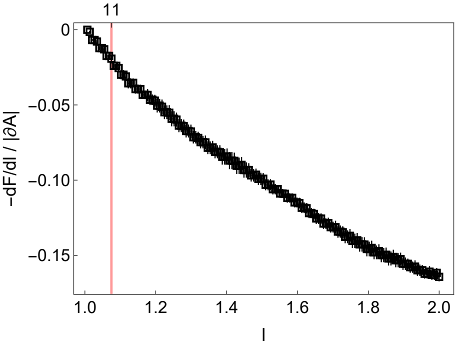

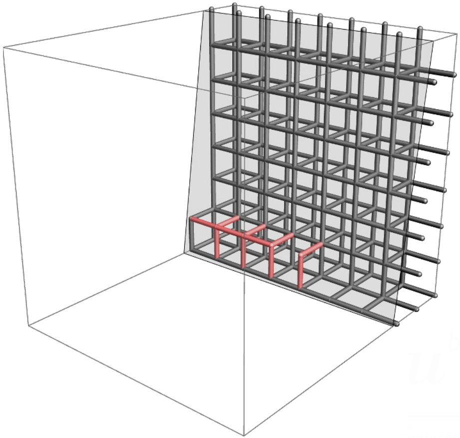

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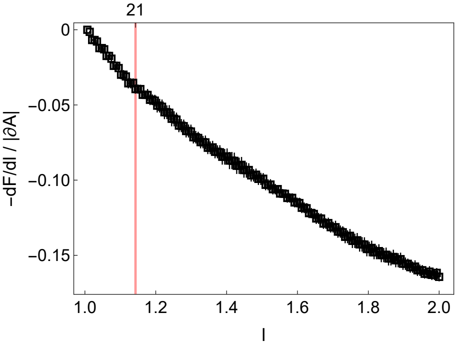

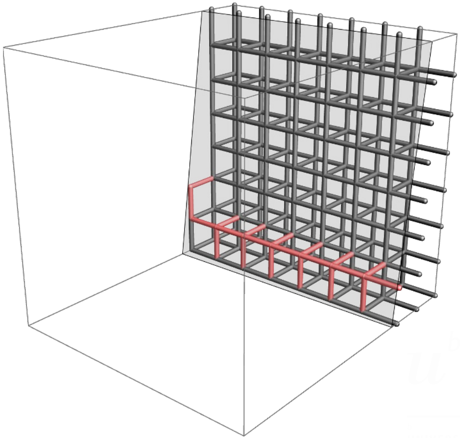

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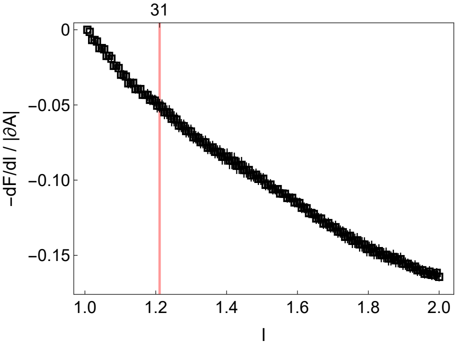

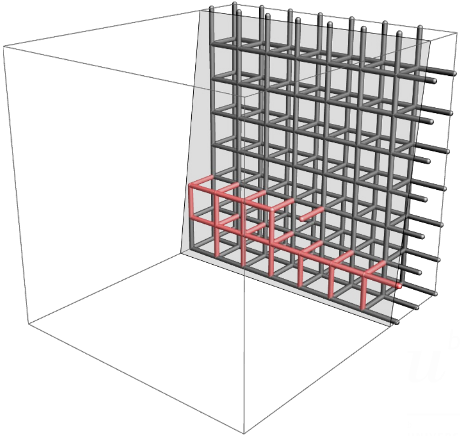

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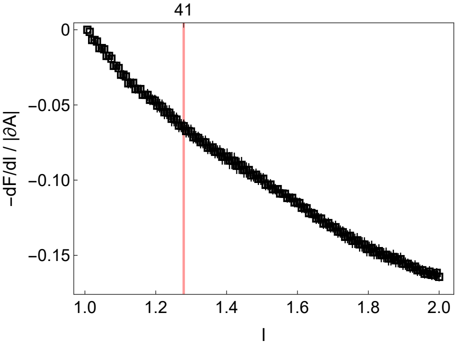

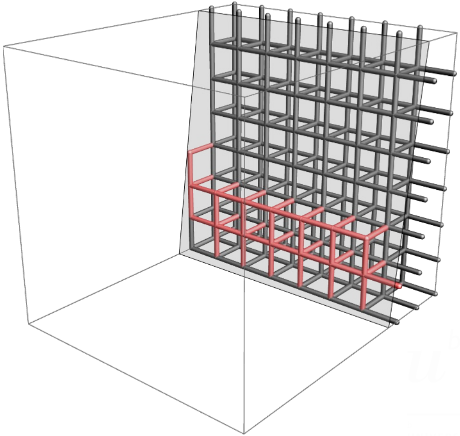

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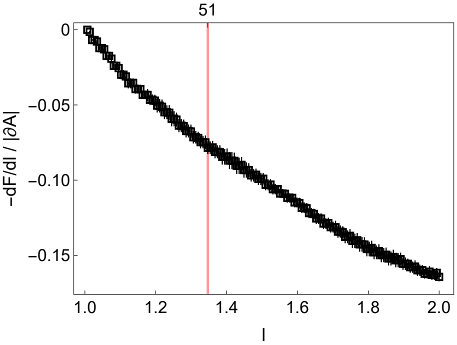

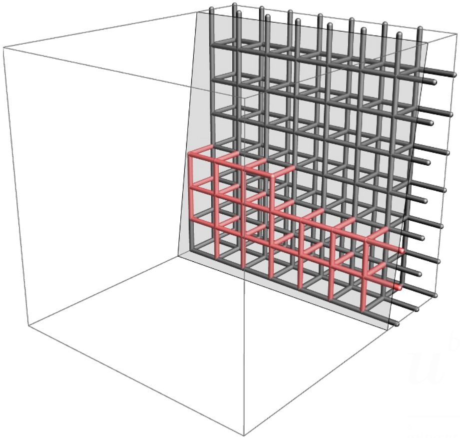

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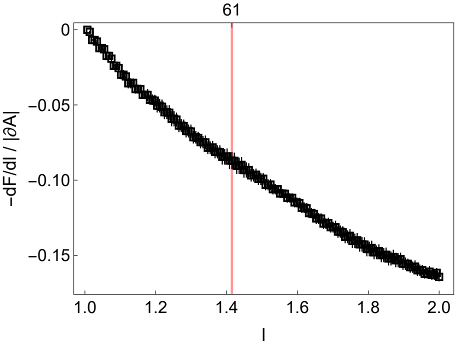

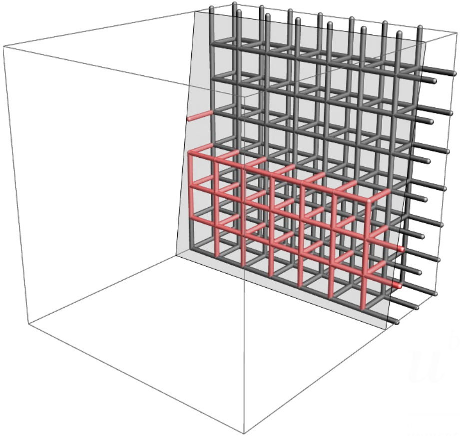

Entangling surface deformation

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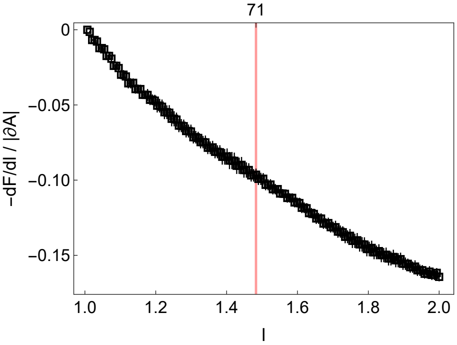

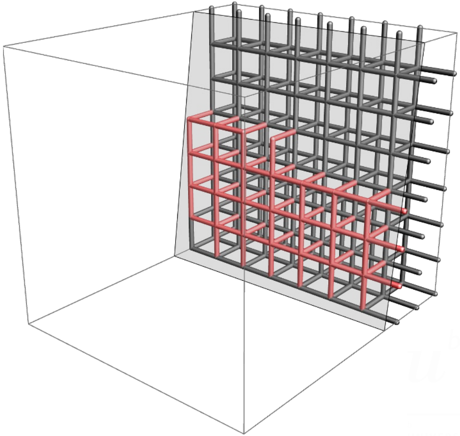

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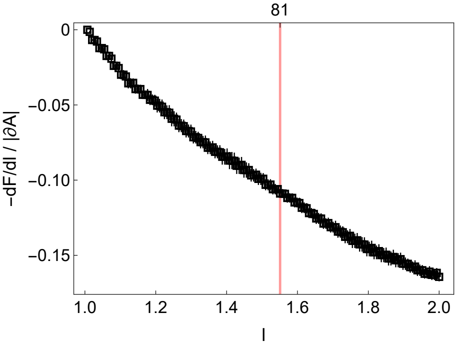

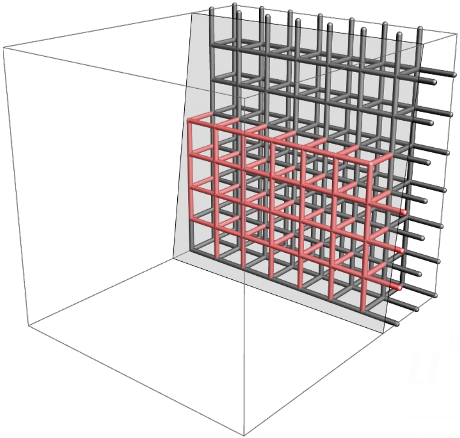

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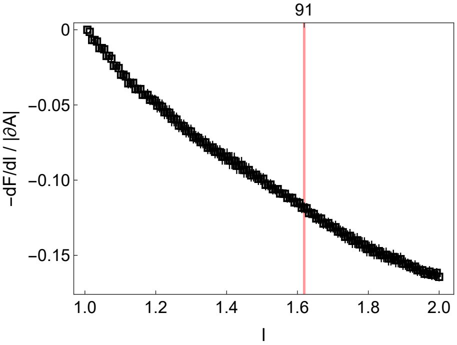

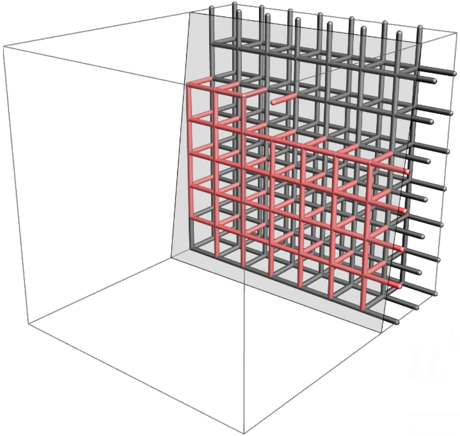

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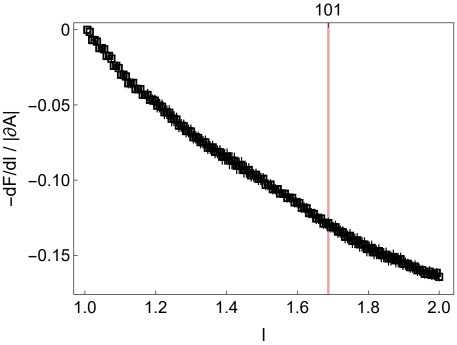

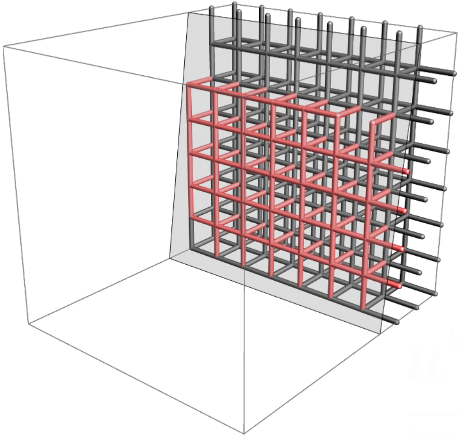

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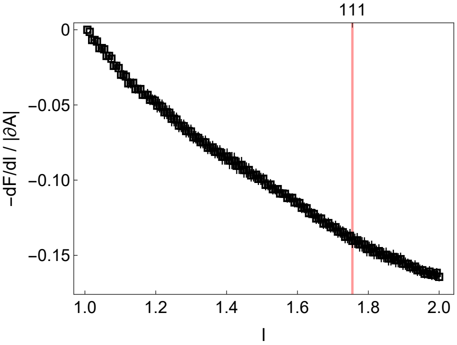

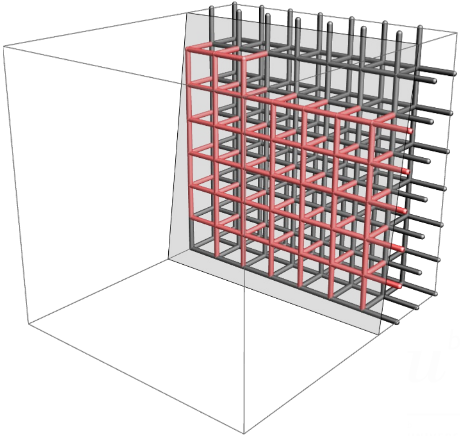

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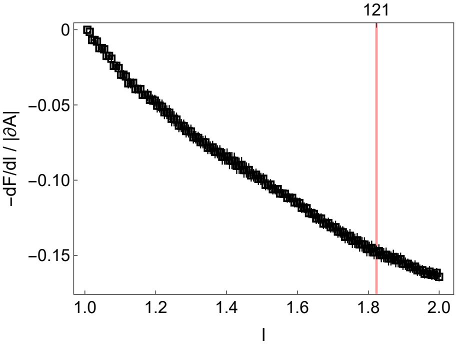

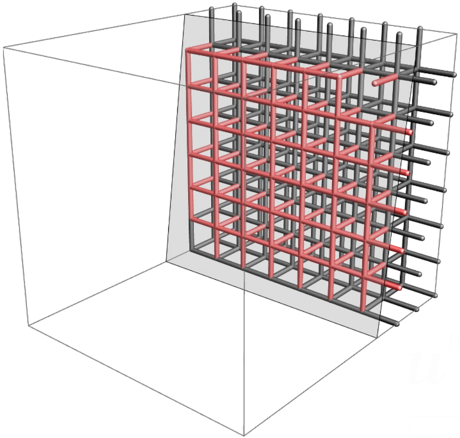

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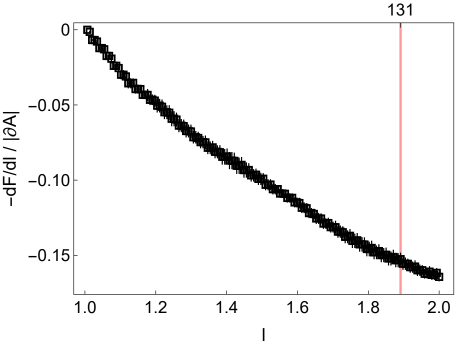

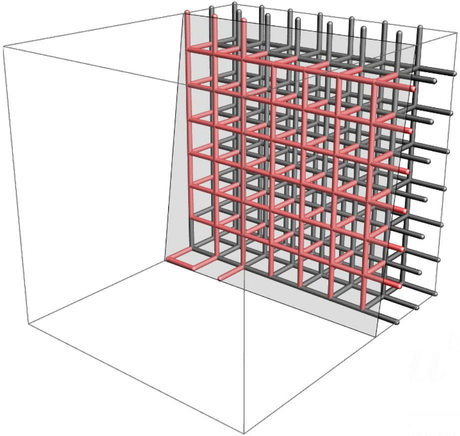

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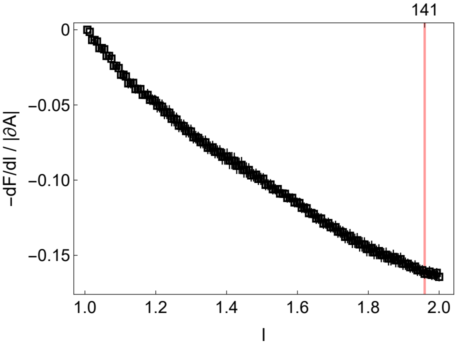

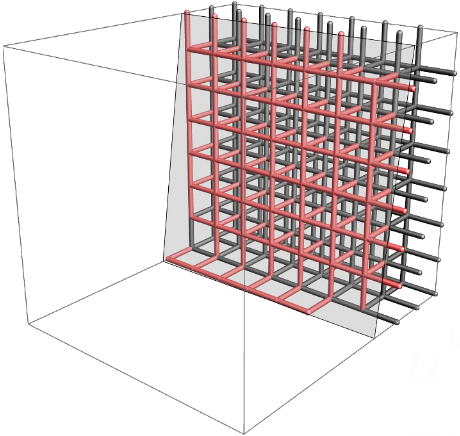

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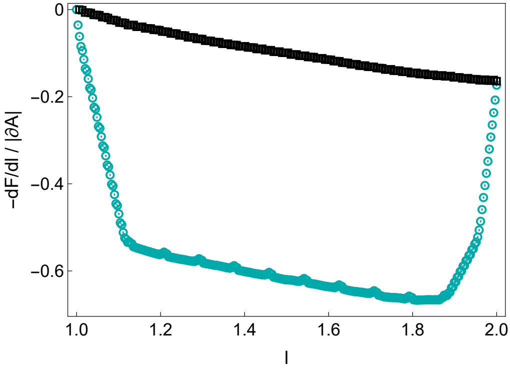
  ➔ SU(5) in (3+1) dimensions:

  comparison of boundary update methods: non-tilted lattice ⟷ tilted lattice

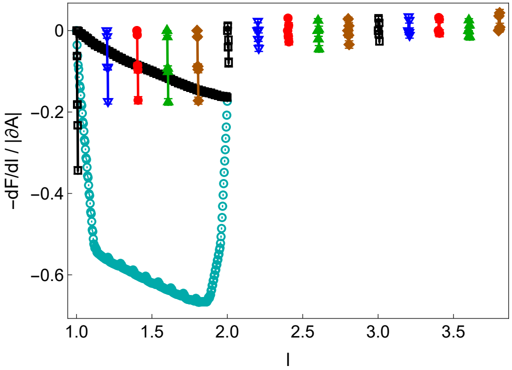


Avoiding remnant free energy barriers

- Tilt lattice with respect to principal directions of "torus"

➔ SU(5) in (3+1) dimensions:

comparison of boundary update methods: non-tilted lattice ⟷ tilted lattice ⟷ local derivative

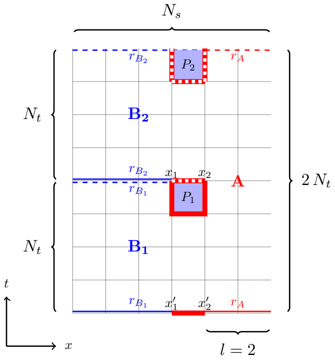


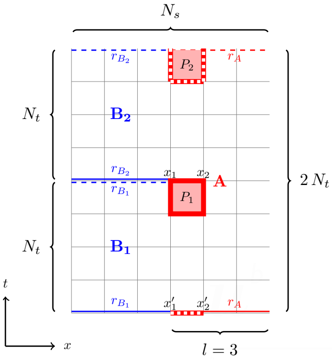
Remaining problems

Single link overlap problem

- BC swap over single non-perpendicular spatial link becomes difficult for $N > 3$

$$p(B \to A) \sim \mathrm{e}^{\frac{\beta}{N}\,\mathrm{Re\,tr}\left(P_{1,A}+P_{2,A}\right)-\frac{\beta}{N}\,\mathrm{Re\,tr}\left(P_{1,B}+P_{2,B}\right)}$$





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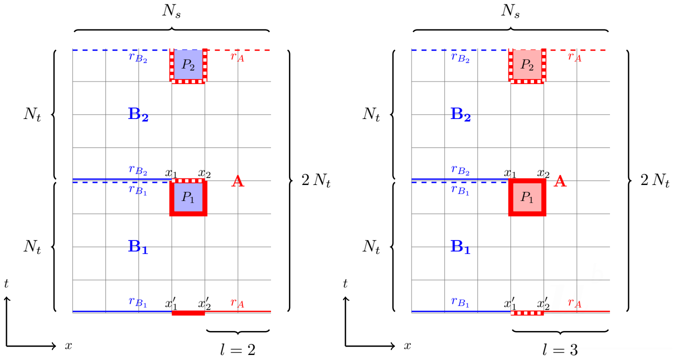
- modified SU(2) sub-group heat-bath update:

  SU(2) $\to p_{\mathrm{acc}} \sim 0.3$

  SU(3) $\to p_{\mathrm{acc}} \sim 0.2$

  SU(5) $\to p_{\mathrm{acc}} \sim 0.005$



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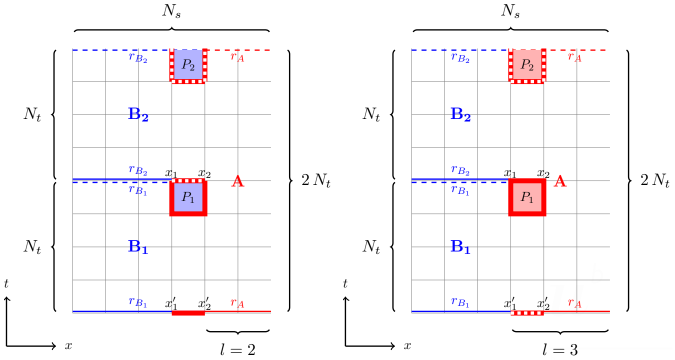
➔ Worm-like update:

  $SU(2) \to p_{acc} \sim 0.45$

  $SU(3) \to p_{acc} \sim 0.35$

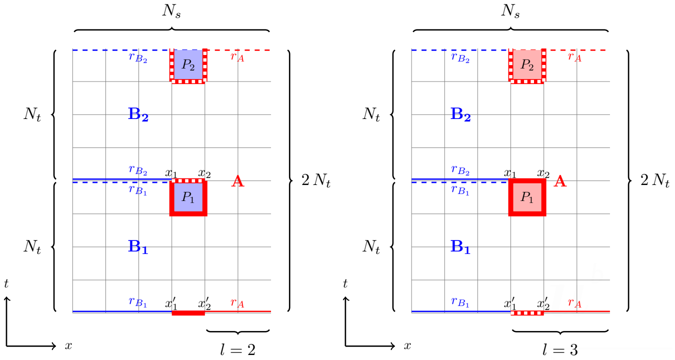
  $SU(5) \to p_{acc} \sim 0.1$



Worm-like BC update

0 pick permutation $\sigma \in \Pi(1,\ldots,s)$, set $i = 1$

while true:



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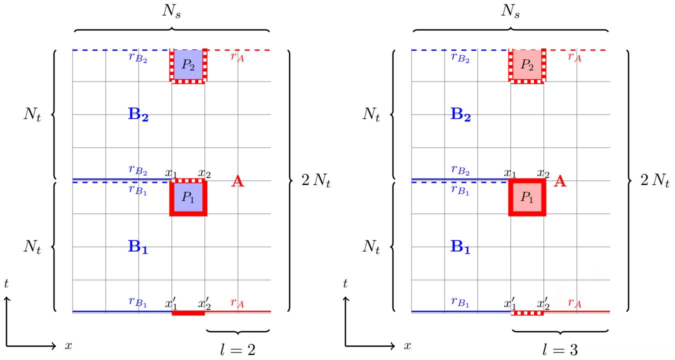
if ($i = 1$ and $\delta i < 0$) or ($i = s$ and $\delta i > 0$): end worm



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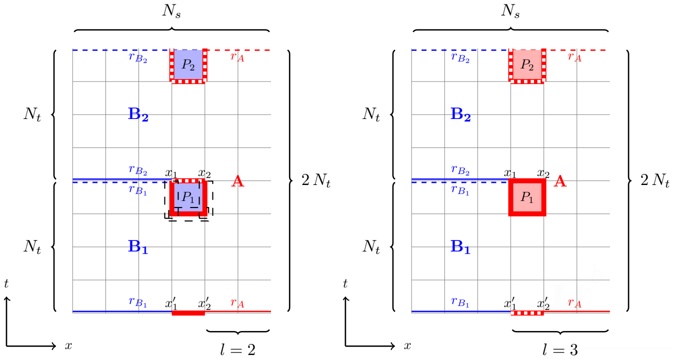

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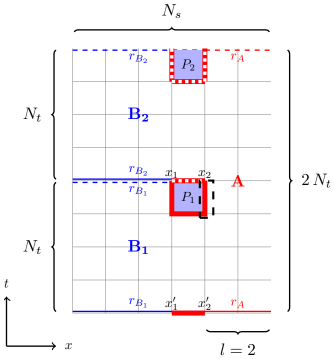

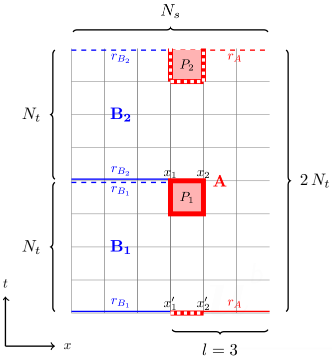

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(one-link int. with Cayley-Hamilton: [TR (2024)])

with probab. $p(\delta i) = \min(1, (Z_A/Z_B)^{\delta i})$:

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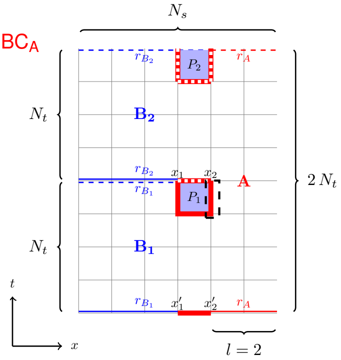

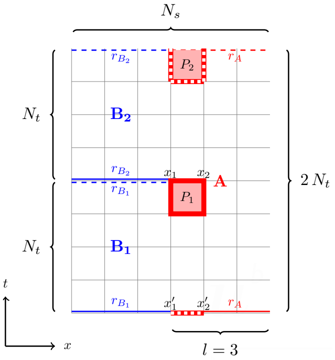

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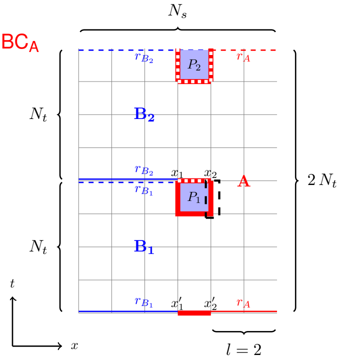

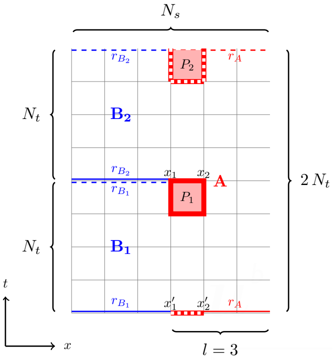

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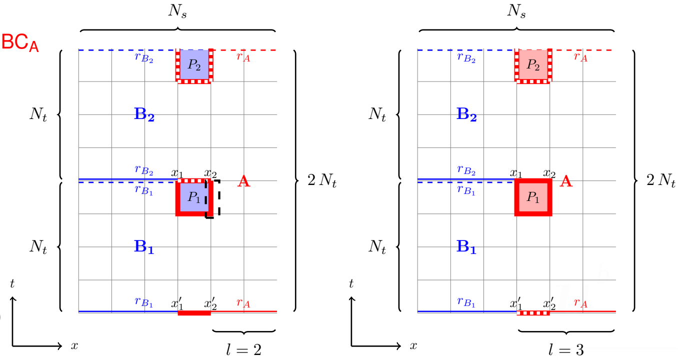
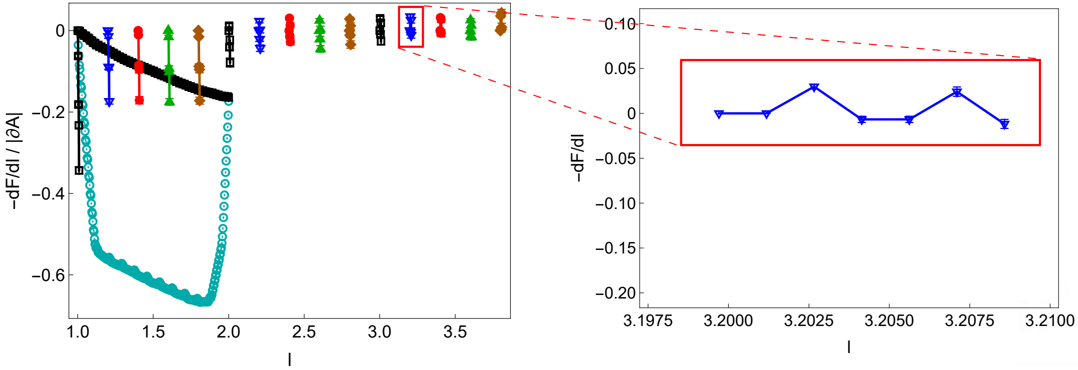
(move choice probab. factors have been omitted)



Remaining problems

Remnant "single cube" free energy barrier?

- For $\ell > 2$ non-monotonic change in free energy during BC change for single spatial cube
  - ➔ auto-correlation issue?
  - ➔ can it be avoided?



Conclusions & outlook

Conclusions

- Entangling surface deformation method with tilted lattice and/or local derivative essentially avoids free energy barriers in determination of entanglement measures (Rényi and entropies) in $SU(N)$ lattice gauge theories.
- Remnant "single cube" free energy barrier can show up for $\ell > 2$.
- Worm-like update for temporal BC flip over spatial link results in higher acceptance rates.

Outlook

- Some ideas to overcome the "single cube" free energy barriers.
- Extended worm-like update over more than one spatial link at a time?
- Applications ...

Thank you!