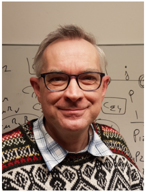
Three-Meson Systems in **Finite & Infinite Volume**

Nordic Lattice Meeting 2024, Lund

Mattias Sjö, CPT Marseille

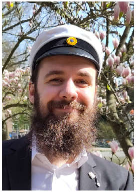
10th June 2024

The collaboration



Hans Bijnens,
Lund U.

Tomáš Husek,
Birmingham U.



Mattias Sjö,
CPT Marseille

Stephen Sharpe,
U. of Washington

Fernando Romero-López,
MIT

Jorge Baeza-Ballesteros,
U. de València

Background

Resonances with 3-body decays

$\omega(782)$ DECAY MODES

	Mode	Fraction (Γ_i/Γ)
Γ_1	$\pi^+\pi^-\pi^0$	(89.2 ± 0.7) %

$a_1(1260)$ DECAY MODES

	Mode	Fraction (Γ_i/Γ)
Γ_1	3π	seen

$N(1440)$ DECAY MODES

The following branching fractions are our estimates, not fits

	Mode	Fraction (Γ_i/Γ)
Γ_1	$N\pi$	55–75 %
Γ_2	$N\eta$	<1 %
Γ_3	$N\pi\pi$	17–50 %

n-body quantization condition

$n \to n$ scattering

Box of size L

n particles

$$\underbrace{\det\left[F_n^{-1} + \mathcal{K}_n\right] = 0}$$

Finite-volume spectrum

n-body quantization condition

$n \to n$ scattering

Box of size L

n particles

$$\det\left[F_n^{-1} + \mathcal{K}_n\right] = 0$$

Finite-volume spectrum

n-body quantization condition

$n \to n$ scattering

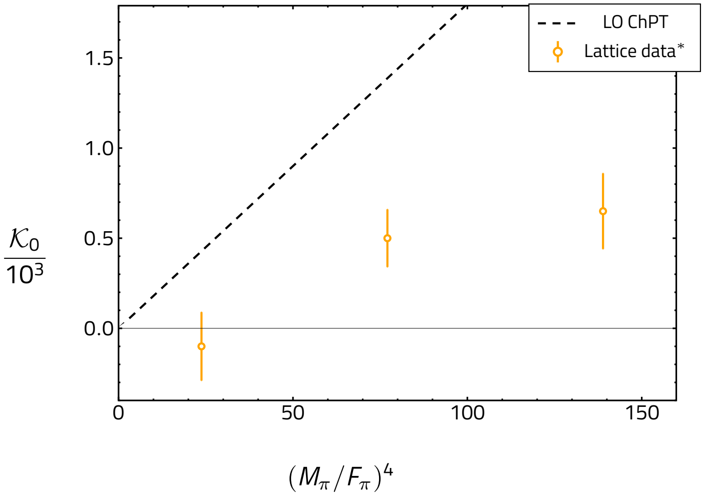
Box of size L

n particles

$$\underbrace{\det\left[F_n^{-1} + \mathcal{K}_n\right] = 0}$$

Finite-volume spectrum

The tension that was



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe,
"*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, 2021.06144[hep-lat]

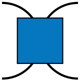
The K-matrix formalism

Hansen & Sharpe, "*Lattice QCD and Three-particle Decays of Resonances*"
Ann.Rev.Nucl.Part.Sci., 1901.00483[hep-lat]

Lüscher, "*Volume Dependence of the Energy Spectrum in Quantum Field Theories*"
Commun.Math.Phys. (1986)

Hansen & Sharpe, "*Relativistic, model independent, three-particle quantization condition*"
Phys.Rev.D, 1408.5933[hep-lat]

(in)finite-volume 2-pt amplitude

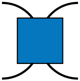
$\mathcal{M}_2 \equiv$ 

- Infinite volume: **integral** over internal momenta
- Finite volume: **sum** over internal momenta

- **Poisson**: sum & integral are **equal** if non-singular (up to exponentially suppressed terms)

- Sum-integral diff. $\Leftrightarrow$ **on-shell** internal momenta (up to exponentially suppressed terms)
- Assume >2 on-shell particles not possible

(in)finite-volume 2-pt amplitude

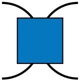
$\mathcal{M}_2 \equiv$ 

- Infinite volume: **integral** over internal momenta
- Finite volume: **sum** over internal momenta

- **Poisson**: sum & integral are **equal** if non-singular (up to exponentially suppressed terms)

- Sum-integral diff. $\Leftrightarrow$ **on-shell** internal momenta (up to exponentially suppressed terms)
- Assume >2 on-shell particles not possible

(in)finite-volume 2-pt amplitude

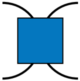
$\mathcal{M}_2 \equiv$ 

- Infinite volume: **integral** over internal momenta
- Finite volume: **sum** over internal momenta

- **Poisson**: sum & integral are **equal** if non-singular
(up to exponentially suppressed terms)

- Sum-integral diff. $\Leftrightarrow$ **on-shell** internal momenta
(up to exponentially suppressed terms)
- Assume >2 on-shell particles not possible

(in)finite-volume 2-pt amplitude

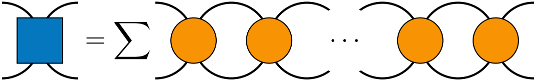
$\mathcal{M}_2 \equiv$ 

- Infinite volume: **integral** over internal momenta
- Finite volume: **sum** over internal momenta

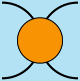
- **Poisson**: sum & integral are **equal** if non-singular
  (up to exponentially suppressed terms)

- Sum-integral diff. $\Leftrightarrow$ **on-shell** internal momenta
  (up to exponentially suppressed terms)
- Assume >2 on-shell particles not possible

...in terms of kernels

$\mathcal{M}_2 \equiv$ 

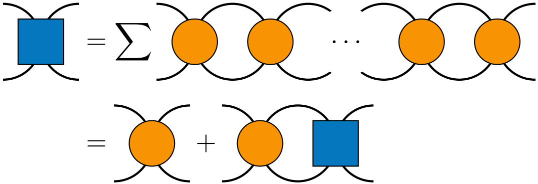
Bethe-Salpeter kernel

$B_2 \equiv$ 

- sum of all 2-particle irreducible diagrams
- is **the same** in both finite and infinite volume
  (up to exponentially suppressed terms)

Recurrence relation

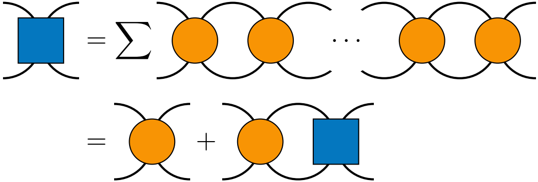
$\mathcal{M}_2 \equiv$



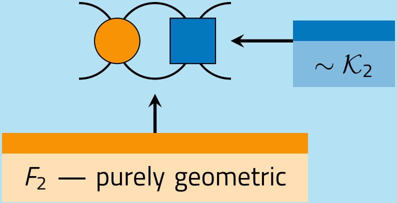
Resummation

Recurrence relation

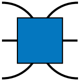
$\mathcal{M}_2 \equiv$



Resummation



On to 3 particles!

$\mathcal{M}_3 \equiv$  $= \dots$

Many more possibilities, some not too complicated:

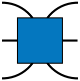
- Chain of B_3:

  Like before, but now with $\boldsymbol{F_3}$

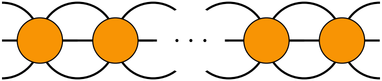
- Chain of B_2:

  Sum into $\mathcal{M}_2$, absorb into F_3 (no longer purely geometric)

On to 3 particles!

$\mathcal{M}_3 \equiv$  $= \dots$

Many more possibilities, some not too complicated:

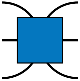
- Chain of B_3: $\dots +$  $+ \dots$

  Like before, but now with $\boldsymbol{F_3}$

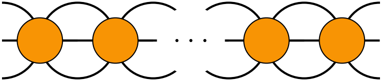
- Chain of B_2: $\dots +$ $+ \dots$

  Sum into $\mathcal{M}_2$, absorb into F_3 (no longer purely geometric)

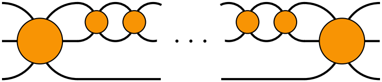
On to 3 particles!

$\mathcal{M}_3 \equiv$  $= \dots$

Many more possibilities, some not too complicated:

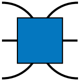
- Chain of B_3: $\dots +$  $+ \dots$

  Like before, but now with $\boldsymbol{F_3}$

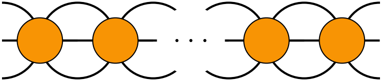
- Chain of B_2: $\dots +$  $+ \dots$

  Sum into $\mathcal{M}_2$, absorb into F_3 (no longer purely geometric)

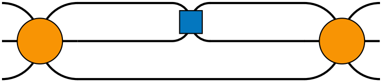
On to 3 particles!

$\mathcal{M}_3 \equiv$  $= \ldots$

Many more possibilities, some not too complicated:

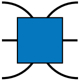
- Chain of B_3: $\ldots +$  $+ \ldots$

  Like before, but now with $\boldsymbol{F_3}$

- Chain of B_2: $\ldots +$  $+ \ldots$

  Sum into $\mathcal{M}_2$, absorb into F_3 (no longer purely geometric)

On to 3 particles!

$\mathcal{M}_3 \equiv$  $= \dots$

Many more possibilities, some **very complicated**:

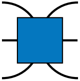
- Alternating $\mathcal{M}_2$'s:

  New matrix $\boldsymbol{G}_\infty$ (more on it later)

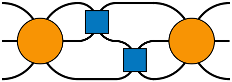
- ...with loops:

  On-shell loop momenta **remain to be integrated**

On to 3 particles!

$\mathcal{M}_3 \equiv$  $= \ldots$

Many more possibilities, some **very complicated**:

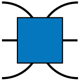
- Alternating $\mathcal{M}_2$'s: $\ldots +$  $+ \ldots$

  New matrix $\boldsymbol{G}_\infty$ (more on it later)

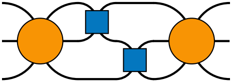
- ...with loops: $\ldots +$ $+ \ldots$

  On-shell loop momenta **remain to be integrated**

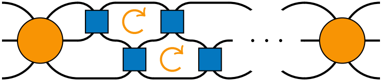
On to 3 particles!

$\mathcal{M}_3 \equiv$  $= \dots$

Many more possibilities, some **very complicated**:

- Alternating $\mathcal{M}_2$'s: $\dots +$  $+ \dots$

  New matrix $\boldsymbol{G}_\infty$ (more on it later)

- ...with loops: $\dots +$  $+ \dots$

  On-shell loop momenta **remain to be integrated**

Anatomy of the K-matrix

Real,
Lorentz invariant,
Scheme dependent

Related to $\mathcal{M}_3$ through **integral equation**

$\mathcal{K}_{\mathrm{df},3}$

Divergence-free
(no poles or cuts)

Anatomy of the K-matrix

Real,
Lorentz invariant,
Scheme dependent

$\mathcal{K}_{\mathrm{df},3}$

Related to $\mathcal{M}_3$ through **integral equation**

Divergence-free
(no poles or cuts)

Anatomy of the K-matrix

Real,
Lorentz invariant,
Scheme dependent

Related to $\mathcal{M}_3$ through **integral equation**

$\mathcal{K}_{\mathrm{df},3}$

Divergence-free
(no poles or cuts)

Anatomy of the K-matrix

Real,
Lorentz invariant,
Scheme dependent

Related to $\mathcal{M}_3$ through **integral equation**

$$\mathcal{K}_{\mathrm{df},3}$$

Divergence-free
(no poles or cuts)

Anatomy of the K-matrix

Real,
Lorentz invariant,
Scheme dependent

Related to $\mathcal{M}_3^{\mathrm{NLO}}$ through **integral equation**

$$\mathcal{K}_{\mathrm{df},3}^{\mathrm{NLO}}$$

Fixed **power counting** gives **algebraic** relation

Divergence-free
(no poles or cuts)

Anatomy of the K-matrix

Real,
Lorentz invariant,
Scheme dependent

Related to $\mathcal{M}_3^{\text{NLO}}$ through **integral equation**

$\mathcal{K}_{\text{df},3}^{\text{NLO}}$

Fixed **power counting** gives **algebraic** relation

Divergence-free
(no poles or cuts)

The stratergy

- Calculate $\mathcal{M}_3^{\text{NLO}}$
- Discard imaginary part
- Subtract divergences

The $3\pi \to 3\pi$ amplitude

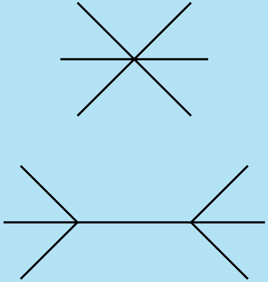
Bijnens & Husek, "*Six-pion amplitude*"
Phys.Rev.D, 2107.06291[hep-ph]

Bijnens, Husek & **Sjö**, "*Six-meson amplitude in QCD-like theories*"
Phys.Rev.D, 2206.14212[hep-ph]

Bijnens, Kampf & **Sjö**, "*Higher-order tree-level amplitudes in the nonlinear sigma model*"
JHEP, 1909.13684[hep-th]

Leading order

Ancient current algebra result



Osborn (1969)

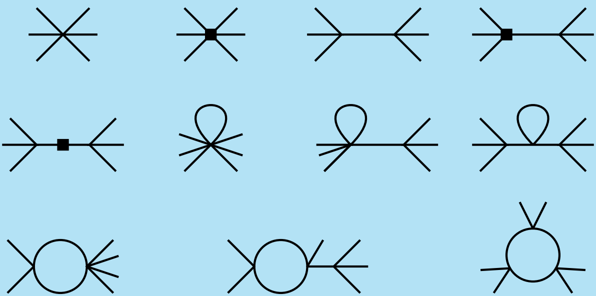
Susskind & Frye (1970)

NLO in CHPT

Vertices

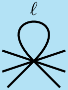
 = LO vertex  = NLO vertex

All the LO and NLO diagrams

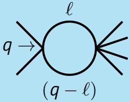


One-Loop Integrals

One- and two-propagator integrals

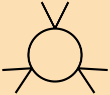


$$\sim \frac{1}{4-d} + \text{(finite)}$$



$$\sim \frac{1}{4-d} + \bar{J}(q^2) + \text{(finite)}$$

Three-propagator integral



$$\sim \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\{1, \ell^\mu, \ell^\mu\ell^\nu, \ell^\mu\ell^\nu\ell^\rho\}}{(\ell^2 - M^2)\left[(\ell - q_1)^2 - M^2\right]\left[(\ell + q_2)^2 - M^2\right]}$$

In principle reducible to $\bar{J}$ — **impractical** — redundant basis instead:

$$\{\bar{J}, C, C_{11}, C_{21}, C_3\}(p_1, \ldots, p_6)$$

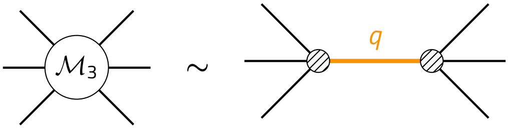
Simplifying the amplitude

$\mathcal{M}_3^{\text{NLO}}$ is a function of...

- 6 particle flavors
- 9 kinematic invariants (8 in $d = 4$)
- 8 free parameters (5 with just pions)
- $\bar{J}(q_i, q_j)$ and 4 $C_X(p_i, p_j, p_k, p_l, p_m, p_n)$'s

$\sim$ **500 pages** in full $\rightarrow$ How to simplify?

Single-particle pole



Factorization

$$\mathcal{M}_3 = \sum_{\substack{\{ijk\} \\ \{lmn\}}} \frac{\mathcal{M}_2(p_i, p_j, p_k, +q) \times \mathcal{M}_2(p_k, p_l, p_n, -q)}{q^2 - M^2 + i\epsilon} + \text{(non-factorizable)}$$

Stripped amplitudes

The 4-point amplitude

$$\begin{aligned}\mathcal{M}^{abcd}(s,t) = &\left[\langle \boldsymbol{abcd}\rangle + \langle dcba\rangle\right] \boldsymbol{B(s,t,u)} + \langle \boldsymbol{ab}\rangle\langle \boldsymbol{cd}\rangle \boldsymbol{C(s,t,u)}\\ &+ \left[\langle acdb\rangle + \langle bdca\rangle\right] B(t,u,s) \quad + \langle ac\rangle\langle bd\rangle C(t,u,s)\\ &+ \left[\langle adbc\rangle + \langle cbda\rangle\right] B(u,s,t) \quad + \langle ad\rangle\langle bc\rangle C(u,s,t)\end{aligned}$$

The *stripped* 4-point amplitude

$$B = \mathcal{M}_{\{4\}}\,, \qquad C = \mathcal{M}_{\{2,2\}}$$

Stripped amplitudes

Flavour structures

$$\mathcal{F}_{\{6\}}(a_1,\dots,a_6) = \langle a_1 \cdots a_6 \rangle$$
$$\mathcal{F}_{\{2,4\}}(a_1,\dots,a_6) = \langle a_1 a_2 \rangle \langle a_3 \cdots a_6 \rangle$$
$$\mathcal{F}_{\{3,3\}}(a_1,\dots,a_6) = \langle a_1 a_2 a_3 \rangle \langle a_4 a_5 a_6 \rangle$$
$$\mathcal{F}_{\{2,2,2\}}(a_1,\dots,a_6) = \langle a_1 a_2 \rangle \langle a_3 a_4 \rangle \langle a_5 a_6 \rangle$$

$$\mathcal{M}(p_1,a_1;p_2,a_2;\dots) = \sum_R \sum_\sigma \mathcal{M}_R(\sigma[p_1,\dots])\mathcal{F}_R(\sigma[a_1,\dots])$$

Stripping

$\sigma \notin$ symmetries of $\mathcal{F}_R$
$\to$ well-known, unique

Deorbiting

$\sigma \in$ symmetries of $\mathcal{F}_R$
$\to$ novel, non-unique!

$\mathcal{M}_3^{\text{NLO}}$ still won't fit on a slide, but not far from it!

Stripped amplitudes

Flavour structures

$$\mathcal{F}_{\{6\}}(a_1,\ldots,a_6) = \langle a_1 \cdots a_6\rangle$$
$$\mathcal{F}_{\{2,4\}}(a_1,\ldots,a_6) = \langle a_1 a_2\rangle\langle a_3 \cdots a_6\rangle$$
$$\mathcal{F}_{\{3,3\}}(a_1,\ldots,a_6) = \langle a_1 a_2 a_3\rangle\langle a_4 a_5 a_6\rangle$$
$$\mathcal{F}_{\{2,2,2\}}(a_1,\ldots,a_6) = \langle a_1 a_2\rangle\langle a_3 a_4\rangle\langle a_5 a_6\rangle$$

$$\mathcal{M}(p_1,a_1;p_2,a_2;\ldots) = \sum_R \sum_\sigma \mathcal{M}_R(\sigma[p_1,\ldots])\mathcal{F}_R(\sigma[a_1,\ldots])$$

Stripping

$\sigma \notin$ symmetries of $\mathcal{F}_R$
$\rightarrow$ well-known, unique

Deorbiting

$\sigma \in$ symmetries of $\mathcal{F}_R$
$\rightarrow$ novel, non-unique!

$\mathcal{M}_3^{\text{NLO}}$ still won't fit on a slide, but not far from it!

Stripped amplitudes

Flavour structures

$$\mathcal{F}_{\{6\}}(a_1,\ldots,a_6) = \langle a_1 \cdots a_6 \rangle$$
$$\mathcal{F}_{\{2,4\}}(a_1,\ldots,a_6) = \langle a_1 a_2 \rangle \langle a_3 \cdots a_6 \rangle$$
$$\mathcal{F}_{\{3,3\}}(a_1,\ldots,a_6) = \langle a_1 a_2 a_3 \rangle \langle a_4 a_5 a_6 \rangle$$
$$\mathcal{F}_{\{2,2,2\}}(a_1,\ldots,a_6) = \langle a_1 a_2 \rangle \langle a_3 a_4 \rangle \langle a_5 a_6 \rangle$$

$$\mathcal{M}(p_1, a_1; p_2, a_2; \ldots) = \sum_R \sum_\sigma \mathcal{M}_R(\sigma[p_1,\ldots])\mathcal{F}_R(\sigma[a_1,\ldots])$$

Stripping

$\sigma \notin$ symmetries of $\mathcal{F}_R$
$\rightarrow$ well-known, unique

Deorbiting

$\sigma \in$ symmetries of $\mathcal{F}_R$
$\rightarrow$ novel, non-unique!

$\mathcal{M}_3^{\mathrm{NLO}}$ still won't fit on a slide, but not far from it!

Calculating the 3-pion K-matrix at NLO

Baeza-Ballesteros, Bijnens, Husek, Romero-López, Sharpe & **Sjö** "*The isospin-3 three-particle K-matrix at NLO in ChPT*"

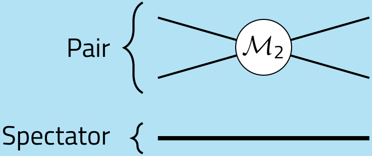
JHEP, 2303.13206[hep-ph]

Baeza-Ballesteros, Bijnens, Husek, Romero-López, Sharpe & **Sjö** "*The three-pion K-matrix at NLO in ChPT*"

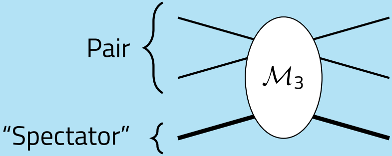
JHEP, 2401.14293[hep-ph]

Building blocks

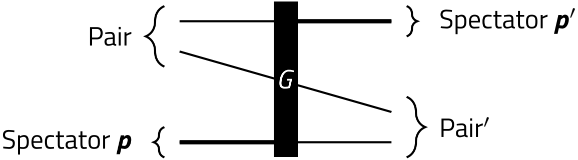
3 particles, 2 scattering



3 particles, 3 scattering



Spectator exchange



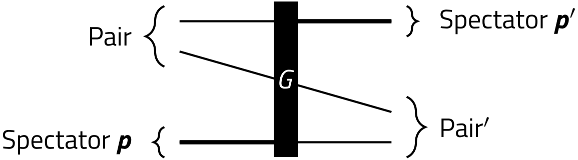
Properties of G

- Purely **on-shell**
- **Propagator-like** near pole:

$$G(\boldsymbol{p}, \boldsymbol{p}')_{lm,l'm'} \sim \frac{1}{(P - p - p')^2 - M^2 + i\epsilon}$$

- Smooth **cutoff** away from pole:
  - No UV problems...
  - ...but **non-analytic**
  - ...and **scheme-dependent**

Spectator exchange



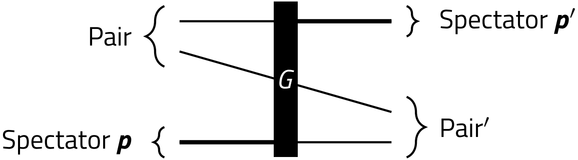
Properties of G

- Purely **on-shell**
- **Propagator-like** near pole:

$$G(\boldsymbol{p}, \boldsymbol{p}')_{lm,l'm'} \sim \frac{1}{(P - p - p')^2 - M^2 + i\epsilon}$$

- Smooth **cutoff** away from pole:
  - No UV problems...
  - ...but **non-analytic**
  - ...and **scheme-dependent**

Spectator exchange



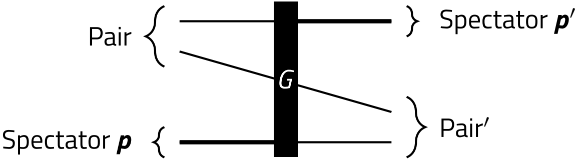
Properties of G

- Purely **on-shell**
- **Propagator-like** near pole:

$$G(\boldsymbol{p}, \boldsymbol{p}')_{lm,l'm'} \sim \frac{1}{(P - p - p')^2 - M^2 + i\epsilon}$$

- Smooth **cutoff** away from pole:
  - No UV problems...
  - ...but **non-analytic**
  - ...and **scheme-dependent**

Spectator exchange



Properties of G

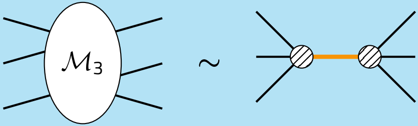
- Purely **on-shell**
- **Propagator-like** near pole:

$$G(\boldsymbol{p}, \boldsymbol{p}')_{lm,l'm'} \sim \frac{1}{(P - p - p')^2 - M^2 + i\epsilon}$$

- Smooth **cutoff** away from pole:
  - No UV problems...
  - ...but **non-analytic**
  - ...and **scheme-dependent**

$\mathcal{K}_{\mathbf{df},3}$ at leading order

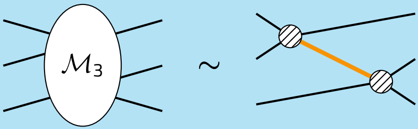
s-channel exchange



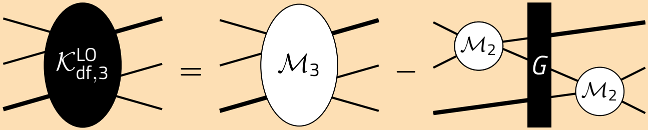
- Only present at **isospin 1**
- **No subtraction** needed since pole is sub-threshold

$\mathcal{K}_{\mathbf{df},3}$ at leading order

One-particle exchange (OPE) pole

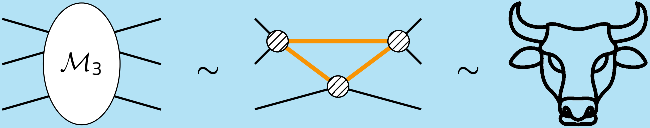


OPE subtraction

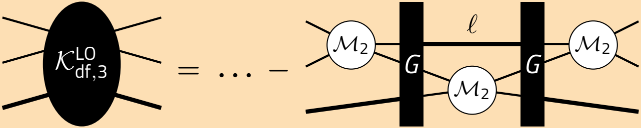


$\mathcal{K}_{\mathrm{df},3}$ at next-to-leading order

Bull's head cut



Bull's head subtraction



The bull's head

The bull's head integral is **awful**:

- Triangle loop ⇒ complicated, pole-ridden integrand
- On-shell ⇒ no loop momentum shift
- Non-analytic ⇒ no Wick rotation, etc.

Different approaches

- Divide & conquer
  simple part with poles + complicated part (numerics-friendly)
- Subtract & conquer
  Cancel divergences against $\mathcal{M}_3$ *before* evaluating
- Brute-force numerics
  Because Tomáš is a Mathematica wizard
- Semi-analytic
  Threshold-expand, then apply deep magic

The bull's head

The bull's head integral is **awful**:

- Triangle loop ⇒ complicated, pole-ridden integrand
- On-shell ⇒ no loop momentum shift
- Non-analytic ⇒ no Wick rotation, etc.

Different approaches

- Divide & conquer
  simple part with poles + complicated part (numerics-friendly)
- Subtract & conquer
  Cancel divergences against $\mathcal{M}_3$ *before* evaluating
- Brute-force numerics
  Because Tomáš is a Mathematica wizard
- Semi-analytic
  Threshold-expand, then apply deep magic

The bull's head

The bull's head integral is **awful**:

- Triangle loop $\Rightarrow$ complicated, pole-ridden integrand
- On-shell $\Rightarrow$ no loop momentum shift
- Non-analytic $\Rightarrow$ no Wick rotation, etc.

Different approaches

- Divide & conquer
  simple part with poles + complicated part (numerics-friendly)
- Subtract & conquer
  Cancel divergences against $\mathcal{M}_3$ *before* evaluating
- Brute-force numerics
  Because Tomáš is a Mathematica wizard
- Semi-analytic
  Threshold-expand, then apply deep magic

The bull's head

The bull's head integral is **awful**:

- Triangle loop ⇒ complicated, pole-ridden integrand
- On-shell ⇒ no loop momentum shift
- Non-analytic ⇒ no Wick rotation, etc.

Different approaches

- Divide & conquer
  simple part with poles + complicated part (numerics-friendly)
- Subtract & conquer
  Cancel divergences against $\mathcal{M}_3$ *before* evaluating
- Brute-force numerics
  Because Tomáš is a Mathematica wizard
- Semi-analytic
  Threshold-expand, then apply deep magic

The bull's head

The bull's head integral is **awful**:

- Triangle loop $\Rightarrow$ complicated, pole-ridden integrand
- On-shell $\Rightarrow$ no loop momentum shift
- Non-analytic $\Rightarrow$ no Wick rotation, etc.

Different approaches

- Divide & conquer
  simple part with poles + complicated part (numerics-friendly)
- Subtract & conquer
  Cancel divergences against $\mathcal{M}_3$ *before* evaluating
- Brute-force numerics
  Because Tomáš is a Mathematica wizard
- Semi-analytic
  Threshold-expand, then apply deep magic

Threshold expansion

Expansion parameters

$$\Delta \propto P^2 - (3M_\pi)^2$$ (**system** above-threshold-ness)

$$\Delta_i^{(\prime)} \propto (P - p_i^{(\prime)})^2 - (2M_\pi)^2$$ (**pair** above-threshold-ness)

$$\tilde{t}_{ij} \propto (p_i - p_j')^2$$ (**spectator** above-threshold-ness)

Compound parameters

$$\Delta_\mathrm{A} = \sum(\Delta_i^2 + \Delta_i'^2) - \Delta^2 \qquad \Delta_\mathrm{B} = \sum \tilde{t}_{ij}^2 - \Delta^2$$

Maximum isospin threshold expansion

$$\mathcal{K}_{\mathrm{df},3}^{[I=3]} = \mathcal{K}_0 + \mathcal{K}_1\Delta + \mathcal{K}_2\Delta^2 + \mathcal{K}_\mathrm{A}\Delta_\mathrm{A} + \mathcal{K}_\mathrm{B}\Delta_\mathrm{B} + \mathcal{O}(\Delta^3)$$

Threshold expansion

Expansion parameters

$$\Delta \propto P^2 - (3M_\pi)^2$$ (**system** above-threshold-ness)

$$\Delta_i^{(\prime)} \propto (P - p_i^{(\prime)})^2 - (2M_\pi)^2$$ (**pair** above-threshold-ness)

$$\tilde{t}_{ij} \propto (p_i - p'_j)^2$$ (**spectator** above-threshold-ness)

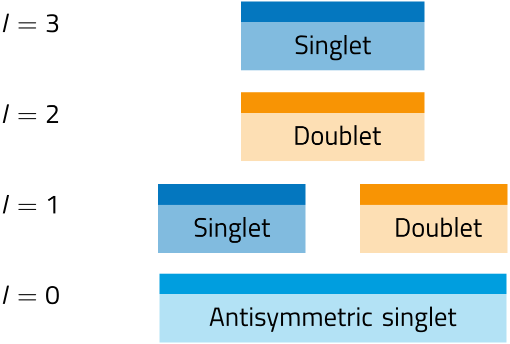
Compound parameters

$$\Delta_\mathrm{A} = \sum(\Delta_i^2 + \Delta_i'^2) - \Delta^2 \qquad \Delta_\mathrm{B} = \sum \tilde{t}_{ij}^2 - \Delta^2$$

Maximum isospin threshold expansion

$$\mathcal{K}_{\mathrm{df},3}^{[I=3]} = \mathcal{K}_0 + \mathcal{K}_1\Delta + \mathcal{K}_2\Delta^2 + \mathcal{K}_\mathrm{A}\Delta_\mathrm{A} + \mathcal{K}_\mathrm{B}\Delta_\mathrm{B} + \mathcal{O}(\Delta^3)$$

Non-maximal isospin



Minimum **isospin threshold expansion**

$$\mathcal{K}_{\mathrm{df},3}^{[I=0]} = \mathcal{K}_0^{\mathrm{AS}} \sum \epsilon_{ijk}\epsilon_{lmn}\tilde{t}_{il}\tilde{t}_{jm} + \mathcal{O}(\Delta^3)$$

Non-maximal isospin

$I = 3$ — Singlet

$I = 2$ — Doublet

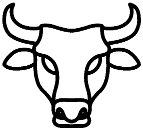
$I = 1$ — Singlet | Doublet

$I = 0$ — Antisymmetric singlet

***Minimum* isospin threshold expansion**

$$\mathcal{K}_{\mathrm{df},3}^{[I=0]} = \mathcal{K}_0^{\mathrm{AS}} \sum \epsilon_{ijk}\epsilon_{lmn}\tilde{t}_{il}\tilde{t}_{jm} + \mathcal{O}(\Delta^3)$$

Semi-analytic evaluation



Expand →

$$\sum \int \mathrm{d}x \,[\text{End-point divergent}](x)$$

↓ **Hadamard partie finie**

$$\sum \int \mathrm{d}x \,[\text{Finite}](x)$$

↓

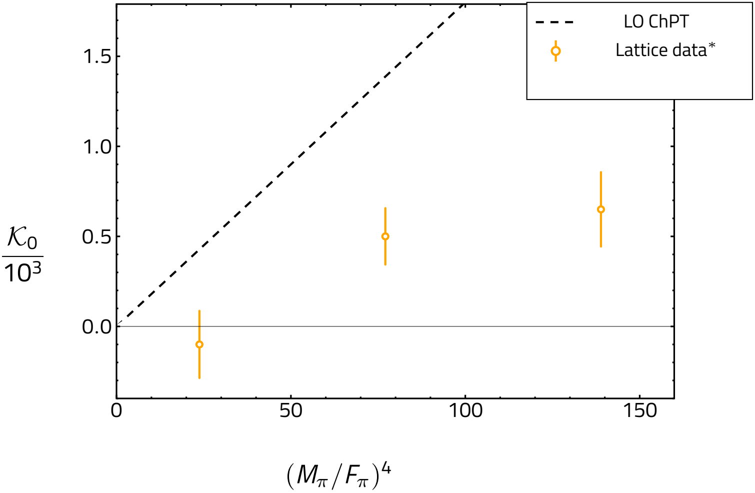
$$\underbrace{\sum \int \mathrm{d}x \,[\text{Simple}](x)}_{\text{analytic}} + \underbrace{\int \mathrm{d}x \,[\text{Small}](x)}_{\substack{\text{numeric} \\ \text{cutoff-dependent}}}$$

Costin & Friedman, "*Foundational aspects of singular integrals*"

J.Functional Analysis, 1401.7045[math.FA]

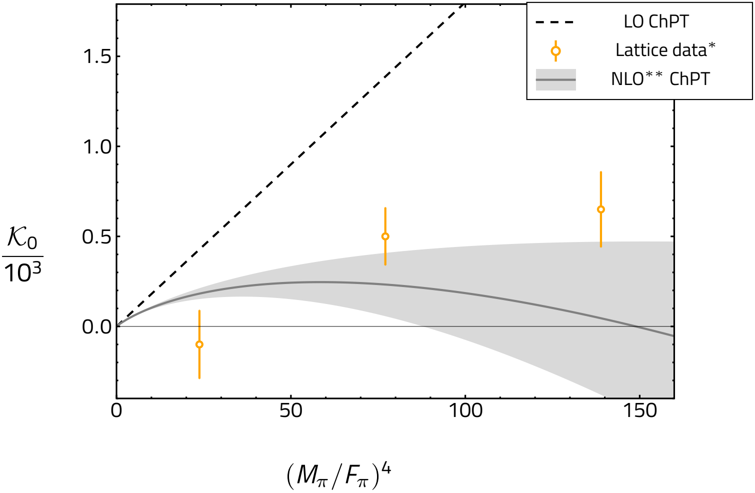
Results

Resolving the tension



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe, *"Three-body interactions from the finite-volume QCD spectrum"* *Phys.Rev.D*, 2021.06144 [hep-lat]

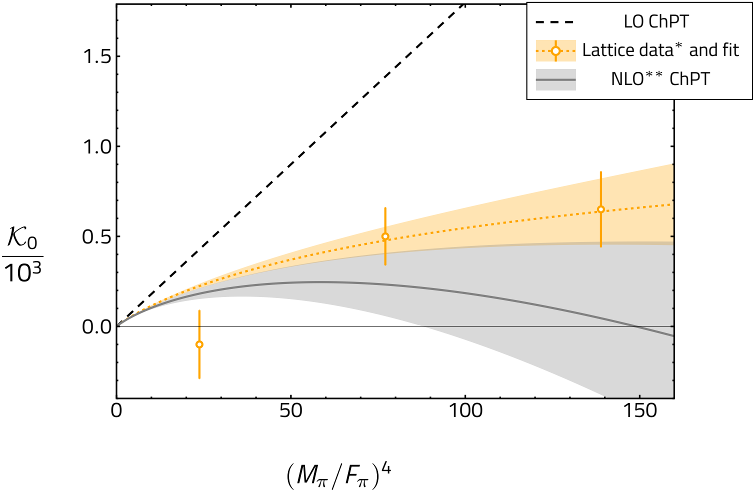
Resolving the tension



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe, "*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, 2021.06144 [hep-lat]

** using LECs from FLAG and Colangelo, Gasser & Leutwyler, "*$\pi\pi$ scattering*" *Nucl.Phys.B*, hep-ph/0103088

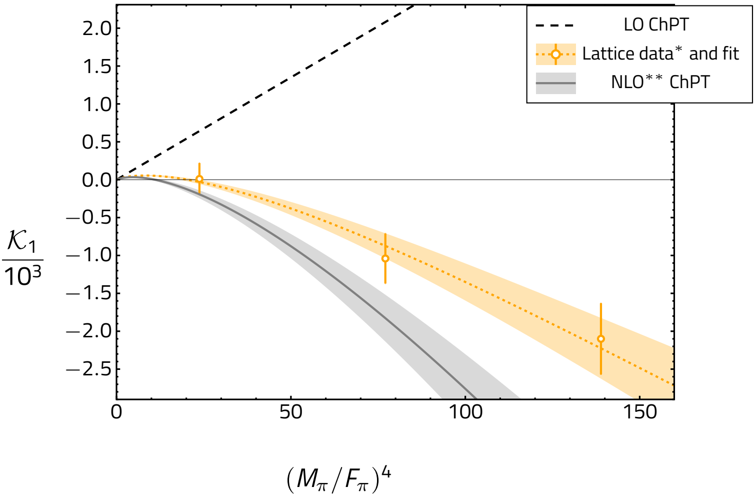
Resolving the tension



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe, "*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, 2021.06144 [hep-lat]

** using LECs from FLAG and Colangelo, Gasser & Leutwyler, "*$\pi\pi$ scattering*" *Nucl.Phys.B*, hep-ph/0103088

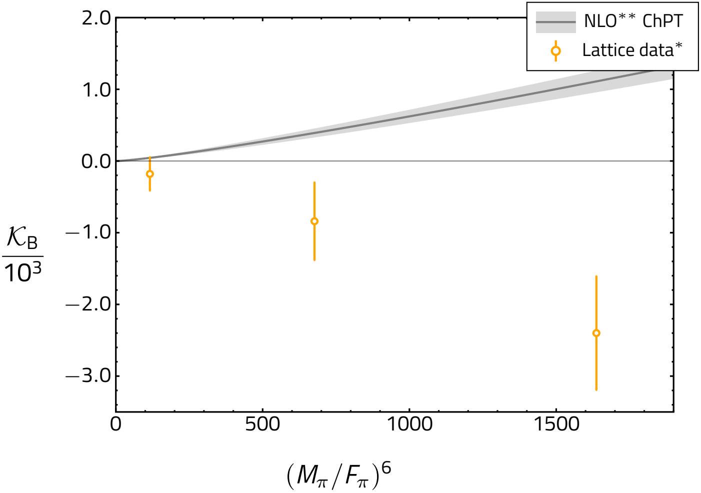
Ditto: Subleading order



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe, "*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, 2021.06144 [hep-lat]

** using LECs from FLAG and Colangelo, Gasser & Leutwyler, "*$\pi\pi$ scattering*" *Nucl.Phys.B*, hep-ph/0103088

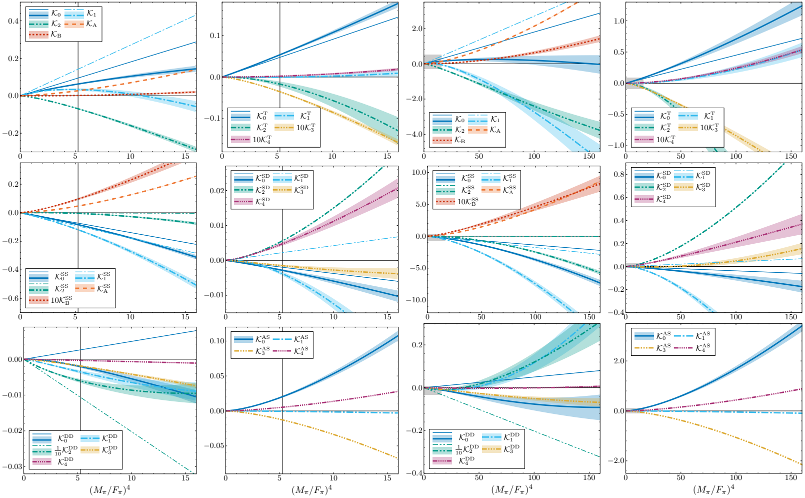
Some tension remains



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe, "*Three-body interactions from the finite-volume QCD spectrum*" — *Phys.Rev.D*, 2021.06144 [hep-lat]

** using LECs from FLAG and Colangelo, Gasser & Leutwyler, "*$\pi\pi$ scattering*" — *Nucl.Phys.B*, hep-ph/0103088

Awaiting more lattice results...



Summary & Outlook

Summary

- All three-pion channels covered
- Main tension resolved
  (where lattice data are available)
- What's next?

Outlook

- Next step: **pion/kaon** systems
  - Lattice data exist:
    Draper, Hanlon, Hörz, Morningstar, Romero-López & Sharpe,
    "*Interactions of πK, $\pi\pi K$ and $KK\pi$ systems at maximal isospin from lattice QCD*"
    JHEP, 2302.13587[hep-lat]
  - ChPT amplitude WIP
- Ultimate goal: **meson/nucleon** systems
  - Groundwork being laid:
    Draper, Hansen, Romero-López & Sharpe,
    "*Three relativistic neutrons in a finite volume*" *JHEP*, 2303.10219[hep-lat]
- Also interesting: analogous $K \to 3\pi$ quantity
  - Formalism in place:
    Hansen, Romero-López & Sharpe,
    "*Decay amplitudes to three hadrons from finite-volume matrix elements*"
    JHEP, 2101.10246[hep-lat]
  - Results in alternative formalism:
    Pang, Bubna, Müller, Rusetsky & Wu,
    "*Lellouch-Lüscher factor for the $K \to 3\pi$ decays*" *JHEP*, 2101.10246[hep-lat]

Outlook

- Next step: **pion/kaon** systems
  - Lattice data exist:
    Draper, Hanlon, Hörz, Morningstar, Romero-López & Sharpe,
    "*Interactions of πK, $\pi\pi K$ and $KK\pi$ systems at maximal isospin from lattice QCD*"
    JHEP, 2302.13587[hep-lat]
  - ChPT amplitude WIP
- Ultimate goal: **meson/nucleon** systems
  - Groundwork being laid:
    Draper, Hansen, Romero-López & Sharpe,
    "*Three relativistic neutrons in a finite volume*" *JHEP*, 2303.10219[hep-lat]
- Also interesting: analogous $K \to 3\pi$ quantity
  - Formalism in place:
    Hansen, Romero-López & Sharpe,
    "*Decay amplitudes to three hadrons from finite-volume matrix elements*"
    JHEP, 2101.10246[hep-lat]
  - Results in alternative formalism:
    Pang, Bubna, Müller, Rusetsky & Wu,
    "*Lellouch-Lüscher factor for the $K \to 3\pi$ decays*" *JHEP*, 2101.10246[hep-lat]

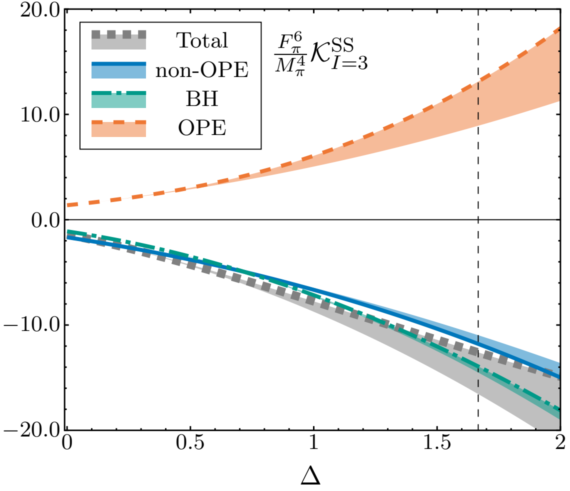
Outlook

- Next step: **pion/kaon** systems
  - Lattice data exist:
    Draper, Hanlon, Hörz, Morningstar, Romero-López & Sharpe,
    "*Interactions of πK, $\pi\pi K$ and $KK\pi$ systems at maximal isospin from lattice QCD*"
    JHEP, 2302.13587[hep-lat]
  - ChPT amplitude WIP
- Ultimate goal: **meson/nucleon** systems
  - Groundwork being laid:
    Draper, Hansen, Romero-López & Sharpe,
    "*Three relativistic neutrons in a finite volume*" *JHEP*, 2303.10219[hep-lat]
- Also interesting: analogous $K \to 3\pi$ quantity
  - Formalism in place:
    Hansen, Romero-López & Sharpe,
    "*Decay amplitudes to three hadrons from finite-volume matrix elements*"
    JHEP, 2101.10246[hep-lat]
  - Results in alternative formalism:
    Pang, Bubna, Müller, Rusetsky & Wu,
    "*Lellouch-Lüscher factor for the $K \to 3\pi$ decays*" *JHEP*, 2101.10246[hep-lat]

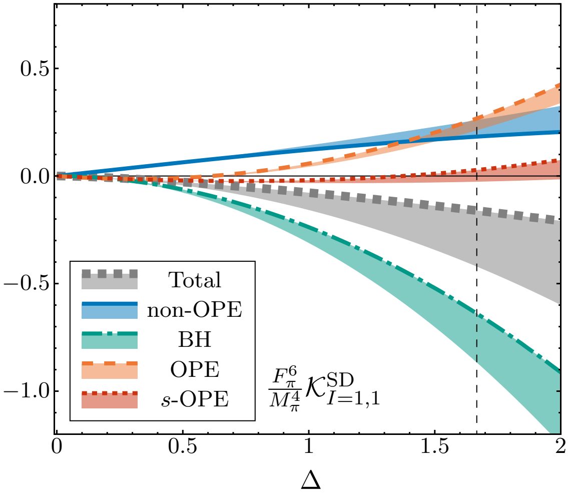
Backup slides

Convergence

The threshold expansion works



...better than it has to



Does ChPT converge?

- Large LO-NLO difference is troubling...
- ...but LO is very constrained
  $\Rightarrow$ **qualitative** difference expected
- Adding NNLO: **extremely difficult**:
  - Two-loop 6-point amplitude
  - Integral relation between $\mathcal{M}_3$ and $\mathcal{K}_{\mathrm{df},3}$

Does ChPT converge?

- Large LO-NLO difference is troubling...
- ...but LO is very constrained
  $\Rightarrow$ **qualitative** difference expected
- Adding NNLO: **extremely difficult**:
  - Two-loop 6-point amplitude
  - Integral relation between $\mathcal{M}_3$ and $\mathcal{K}_{\text{df},3}$

Does ChPT converge?

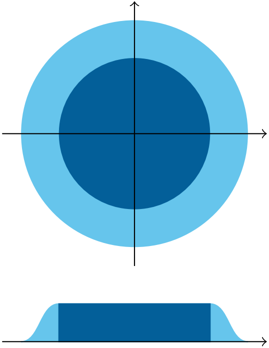
- Large LO-NLO difference is troubling...
- ...but LO is very constrained
  $\Rightarrow$ **qualitative** difference expected
- Adding NNLO: **extremely difficult**:
  - Two-loop 6-point amplitude
  - Integral relation between $\mathcal{M}_3$ and $\mathcal{K}_{\mathrm{df},3}$

More on the bull’s head

Divide & conquer the bull's head

The integral

$$\int \frac{d^3 \boldsymbol{r}}{2\omega_r} \frac{[\text{Non-analytic}]}{[\text{Complicated}]}$$



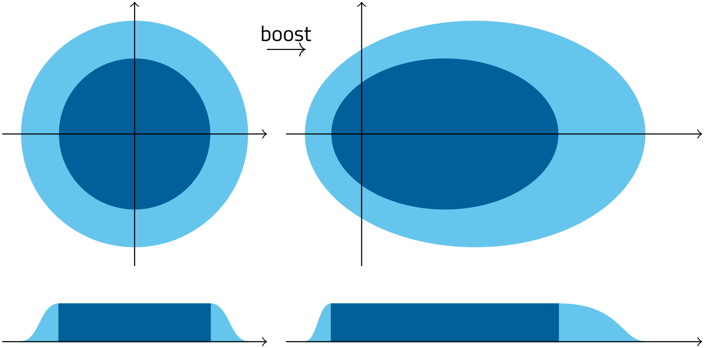
analytic, **has poles**

non-analytic, smooth

Divide & conquer the bull's head

The integral

$$\int \frac{d^3\boldsymbol{r}}{2\omega_r} \frac{\text{[Complicated angular dependence]}}{\text{[Much simpler]}}$$



Divide & conquer the bull's head

The integral

$$\int \frac{d^3\boldsymbol{r}}{2\omega_r}\,\frac{[\text{Simple}] - [\text{Numerics-friendly}]}{[\text{Much simpler}]}$$