$\mathrm{QED_r}$: An infrared-improved finite-volume prescription

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- Precision tests of the Standard Model
- Flavour physics: Cabibbo-Kobayashi-Maskawa (CKM) matrix

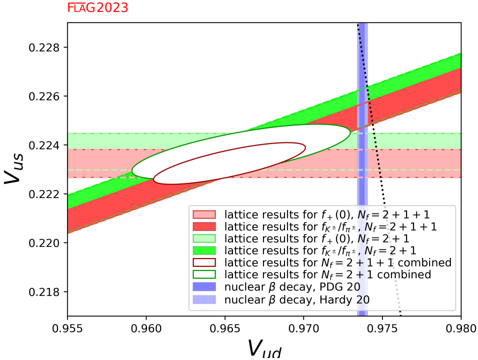
$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

- Fundamental parameters of the Standard Model: Essential

A. Phase of matrix: CP violation and matter asymmetry of the Universe (Jarlskog invariant)

B. Search for new physics by testing unitarity:

$$1 \overset{\mathrm{SM}}{=} |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2$$



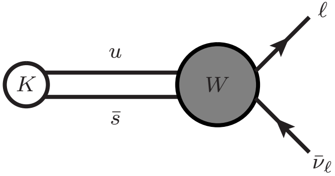
- Tensions with unitarity?
- 1.7–5 σ
- Depends on the input!
- Need improved control of experimental and theoretical uncertainties

- Precision goal: (Sub-)% level
- Isospin-breaking effects crucial:

$$\text{QED: } \alpha \neq 0$$
$$\text{QCD: } m_u \neq m_d$$

- Unknown for horizontal

- Access $|V_{us}|/|V_{ud}|$ from leptonic kaon/pion decays



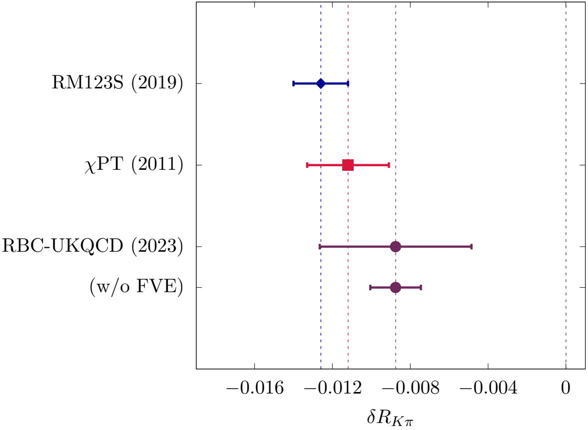
$$\underbrace{\Gamma\left(K^- \to \mu^- \nu_\mu\right)}_{\text{Exp.}} \propto |V_{us}|^2 \underbrace{\frac{\left(m_K^2 - m_\mu^2\right)^2}{m_K^3}}_{\text{Exp.}} \underbrace{f_K^2\left(1+\delta R_K\right)}_{\text{Theory}}$$

- Combine experiment and theory (lattice)

$$\frac{|V_{us}|^2}{|V_{ud}|^2} = \frac{\text{Kaon exp.}}{\text{Pion exp.}} \times \frac{f_K^2}{f_\pi^2}\left(1+\delta R_K - \delta R_\pi\right)$$

- Isospin-breaking corrections in $\delta R_K - \delta R_\pi$: % level precision

[RM123S 2019; Di Carlo, Hansen, NHT, Portelli 2022; RBC/UKQCD 2023]



RBC-UKQCD 23:

$\delta R_{K\pi} = -0.0086(39)$

total	(39)
total (w/o FVE)	(13)
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QED quenching	(5)
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χPT : $\delta R_{K\pi} = -0.0112(21)$

RM123S 19: $\delta R_{K\pi} = -0.0126(14)$

Issue: QED finite-volume effects not sufficiently understood

- Finite-volume $\mathbb{R} \times L^3$: $\mathrm{QED_L}$. Leptonic decay $P \to \ell\nu$

$$Y^{(3)}(L) = \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + \frac{c_3 - 2\,c_3(\mathbf{v}_\ell)}{2\pi} - 2\,A_1(\mathbf{v}_\ell) + 2\log\left(\frac{m_W L}{4\pi}\right)$$
$$- 2\,A_1(\mathbf{v}_\ell)\left[\log\left(\frac{m_P L}{4\pi}\right) + \log\left(\frac{m_\ell L}{4\pi}\right)\right] - \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\,c_2 - 4\,r_\ell^2\,c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right]$$
$$+ \frac{1}{(m_P L)^2}\left[-\frac{F_A^P}{f_P}\,\frac{4\pi\,m_P\,\left[(1+r_\ell^2)^2\,c_1 - 4\,r_\ell^2\,c_1(\mathbf{v}_\ell)\right]}{1-r_\ell^4} + \frac{8\pi\,\left[(1+r_\ell^2)\,c_1 - 2\,c_1(\mathbf{v}_\ell)\right]}{(1-r_\ell^4)}\right]$$
$$+ \frac{32\pi^2\,m_P}{f_P(1-r_\ell^4)\,(m_P L)^3}\left\{c_0(\mathbf{v}_\ell)\,\left[F_V^P - F_A^P + 2m_P^2 r_\ell^2\,A^{(0,1)}\left(0,-m_P^2\right)\right] + c_0\,\mathcal{C}_\ell\right\}$$

- Coefficients: $c_j(\mathbf{v}_\ell)$, c_j, lepton velocity $\mathbf{v}_\ell$
- Structure-dependent form factors: F_V^P, F_A^P, $A^{(0,1)}$
  Lattice [RM-123/Soton 20/22/23, RBC/UKQCD 23], ChPT [Bijnens et al. 92], experiment [...].
- Only know the point-like part of $\mathcal{C}_\ell$ (big): Drives uncertainty

- Clearly we need a better understanding of finite-volume effects
- Gauss' law: Difficult to define charged states in finite volume with periodic boundary conditions

$$Q = \int_V d^3\mathbf{x}\, \nabla \cdot \mathbf{E}(t, \mathbf{x}) = \int_{\partial V} d\mathbf{S} \cdot \mathbf{E} = 0$$

- Related to the photon propagator:

$$D_{\mu\nu}(k) = \frac{\delta_{\mu\nu}}{k^2}$$

- Problem: Photon zero-momentum modes and absence of mass gap
- Need to define QED in a finite volume

QED in a finite volume

- Several prescriptions

  1. $\mathrm{QED_M}$: Photon mass m_γ
     [Endres, Shindler, Tiburzi, Walker-Loud 2016; Bussone, Della Morte, Janowski 2018]
  2. $\mathrm{QED_\infty}$: Do the QED part in infinite volume
     [Feng, Jin 2018; Christ, Feng, Jin, Sachrajda, Wang 2023]
  3. $\mathrm{QED_C}$: Charge-conjugated boundary conditions
     [Kronfeld, Wiese 1991–1993; RC* 2019]
  4. $\mathrm{QED_L^{IR}}$: Exclude/redistribute photon zero-mode
     [Davoudi, Harrison, Jüttner, Portelli, Savage 2019]
     - $\mathrm{QED_L}$: Exclude photon zero-mode [Hayakawa, Uno 2008]
     - $\mathrm{QED_r}$: Redistribute photon zero-mode [Di Carlo, Hansen, NHT, Portelli in prep.]

- Each has advantages/challenges

- Photon momentum $k = (k_0, \mathbf{k})$. Spacetime $\mathbb{R} \times L^3$
- 3-momentum discretised by finite volume extent $\mathbf{k} = \frac{2\pi \mathbf{n}}{L}$:

$$\mathrm{QED}_\mathrm{L}^\mathrm{IR} : \mathrm{QED}_\mathrm{r}, \mathrm{QED}_\mathrm{L} : \quad \Pi = \left\{ \mathbf{n} \in \mathbb{Z}^3 \setminus \{0,0,0\} \right\}$$

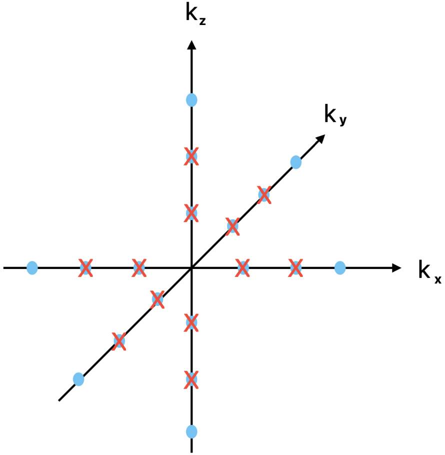
- Allows definition of finite-volume QED
- General form of propagator here: Prescription dependent

$$D_{\mu\nu}(k) = \delta_{\mu\nu} \frac{1 + w_{|\mathbf{n}|^2}}{k^2}$$

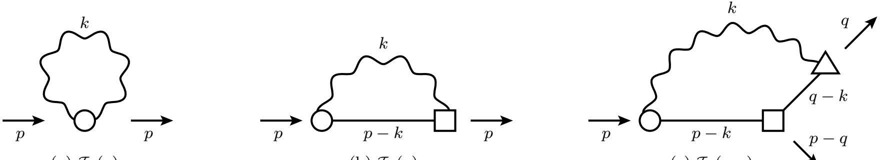
- Weights $w_{|\mathbf{n}|^2}$ in action [Davoudi, Harrison, Jüttner, Portelli, Savage 2019]
- $\mathrm{QED}_\mathrm{L}^\mathrm{IR}$ and QED_r non-zero for finite number of $\mathbf{n} = \mathbf{k}\, L/(2\pi)$
- QED_L $w_{|\mathbf{n}|^2} = 0$

What does this mean?

- $\mathbf{k} = \frac{2\pi \mathbf{n}}{L}$ where $\mathbf{n} \neq \mathbf{0}$
- Add weights $w^2_{|\mathbf{n}|}$ on finite number of modes
- Example: Inner two shells
- $w^2_{|\mathbf{n}|=1}$ and $w^2_{|\mathbf{n}|=2}$
- Can we use the freedom to our advantage?



- Consider observable $\mathcal{O}$ at order α, with photon momentum $k = (k_0, \mathbf{k})$
- Example: Diagram with 2 propagators



(a) $\mathcal{I}_1(p)$ (b) $\mathcal{I}_2(p)$ (c) $\mathcal{I}_3(p,q)$

- Finite-size effects in $\mathcal{O}(L)$ given by:

$$\Delta\mathcal{O}(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{dk_0}{2\pi} \frac{f_{\mathcal{O}}\left(k_0, |\mathbf{k}|, \mathbf{p}\cdot\mathbf{k}\right)}{[(p-k)^2 + m^2]} \frac{1 + w_{|\mathbf{n}|^2}}{k^2}$$

- The function $f_{\mathcal{O}}\left(k_0, |\mathbf{k}|, \mathbf{p}\cdot\mathbf{k}\right)$ has no poles: Depends on observable
- k_0 integral: gives rise to the problematic $c_0\,\mathcal{C}_\ell/L^3$ term

$$\Delta\mathcal{O}(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \left(1 + w_{|\mathbf{n}|^2}\right) g_{\mathcal{O}}(\mathbf{k})$$

- All that remains, where $\mathbf{k} = 2\pi\mathbf{n}/L$:

$$\left(\frac{1}{L^3} \sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \left(1 + w_{|\mathbf{n}|^2}\right) \, g_{\mathcal{O}}(\mathbf{k})$$

- Momentum: $p = (i\omega(\mathbf{p}), \mathbf{p})$, $p^2 = -m^2$. Velocity $\mathbf{v} = \mathbf{p}/\omega(\mathbf{p})$
- Expand in L: Define coefficients

$$c_j^w(\mathbf{v}) = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int d^3\mathbf{n} \right) \frac{1 + w_{|\mathbf{n}|^2}}{|\mathbf{n}|^j \left(1 - \mathbf{v}\cdot\hat{\mathbf{n}}\right)}$$

- Note in particular that

$$c_j^w(\mathbf{v}) = \underbrace{c_j(\mathbf{v})}_{\mathrm{QED_L}} + \sum_{\mathbf{n}\neq\mathbf{0}} \frac{w_{|\mathbf{n}|^2}}{|\mathbf{n}|^j \left(1 - \mathbf{v}\cdot\hat{\mathbf{n}}\right)}$$

- New formula in $\mathrm{QED}_{\mathrm{L}}^{\mathrm{IR}}$ is

$$
\begin{aligned}
Y^{(3)}(L) = \; & \frac{3}{4} + 4 \log\left(\frac{m_\ell}{m_W}\right) + \frac{c_3^w - 2\, c_3^w(\mathbf{v}_\ell)}{2\pi} - 2\, A_1(\mathbf{v}_\ell) + 2 \log\left(\frac{m_W L}{4\pi}\right) \\
& - 2\, A_1(\mathbf{v}_\ell)\left[\log\left(\frac{m_P L}{4\pi}\right) + \log\left(\frac{m_\ell L}{4\pi}\right)\right] - \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\, c_2^w - 4\, r_\ell^2\, c_2^w(\mathbf{v}_\ell)}{1 - r_\ell^4}\right] \\
& + \frac{1}{(m_P L)^2}\left[- \frac{F_A^P}{f_P} \frac{4\pi\, m_P\, \left[(1+r_\ell^2)^2\, c_1^w - 4\, r_\ell^2\, c_1^w(\mathbf{v}_\ell)\right]}{1 - r_\ell^4} + \frac{8\pi\, \left[(1+r_\ell^2)\, c_1^w - 2\, c_1^{\mathrm{w}}(\mathbf{v}_\ell)\right]}{(1 - r_\ell^4)}\right] \\
& + \frac{32\pi^2\, m_P}{f_P(1 - r_\ell^4)\,(m_P L)^3}\left\{ c_0^w(\mathbf{v}_\ell)\, \left[F_V^P - F_A^P + 2 m_P^2 r_\ell^2\, A^{(0,1)}\left(0, -m_P^2\right)\right] + c_0^w\, \mathcal{C}_\ell \right\}
\end{aligned}
$$

- So far we did not specify the weights $w_{|\mathbf{n}|^2}$
- Problematic part is $c_0^w\, \mathcal{C}_\ell$
- Can we choose the weights cleverly?

Minimal choice: $\mathrm{QED_r}$

- Want $c_0^w = 0$

$$c_0^w = \underbrace{c_0}_{=-1} + \sum_{|\mathbf{n}|} w_{|\mathbf{n}|^2}$$

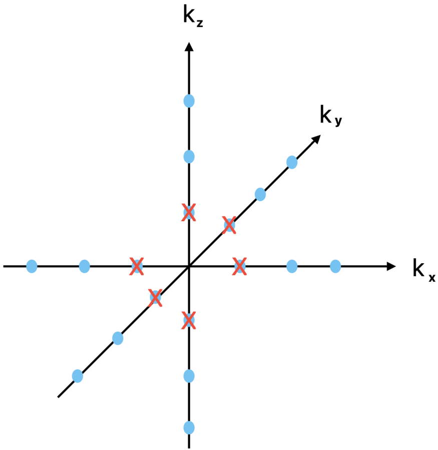
- Solution: $w_{|\mathbf{n}|^2} = \delta_{|\mathbf{n}|,1}/6$

$$\overline{c}_0 = 0$$

- Why it works: $|\mathbf{k}| \propto 1/L$

- Generally:

$$\overline{c}_j(\mathbf{v}) = \underbrace{c_j(\mathbf{v})}_{\mathrm{QED_L}} + \frac{1}{6} \sum_{|\mathbf{n}|=1} \frac{1}{(1-\mathbf{v}\cdot\hat{\mathbf{n}})}$$



- Our formula is now $(\mathrm{QED_r})$:

$$
\begin{aligned}
Y^{(3)}(L) = \; & \frac{3}{4} + 4\,\log\left(\frac{m_\ell}{m_W}\right) + \frac{\bar{c}_3 - 2\,\bar{c}_3(\mathbf{v}_\ell)}{2\pi} - 2\,A_1(\mathbf{v}_\ell) + 2\,\log\left(\frac{m_W L}{4\pi}\right) \\
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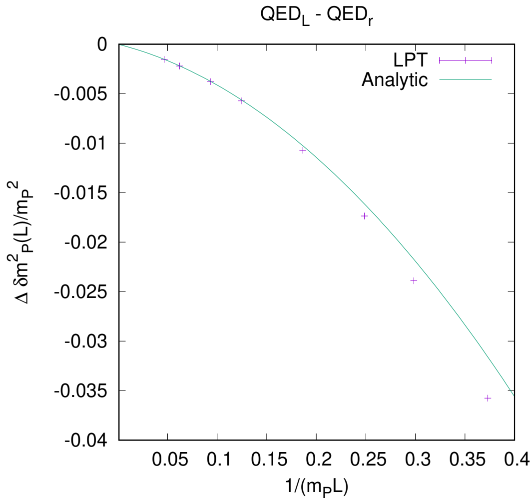
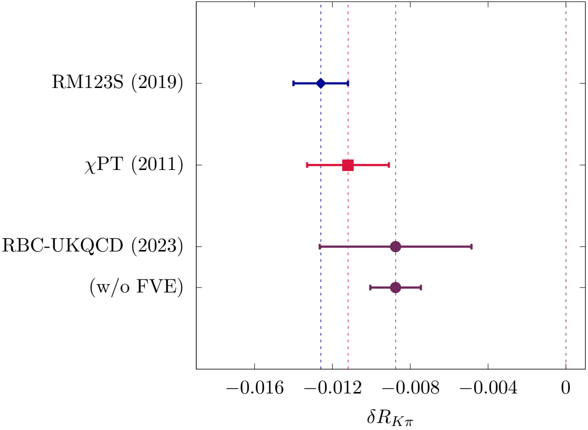
& + \frac{32\pi^2\,m_P}{f_P(1-r_\ell^4)\,(m_P L)^3}\,\bar{c}_0(\mathbf{v}_\ell)\,\left[F_V^P - F_A^P + 2m_P^2 r_\ell^2\,A^{(0,1)}\left(0,-m_P^2\right)\right]
\end{aligned}
$$

- Our problem is solved!
- Current study: Asymptotic behaviour of $\frac{\bar{c}_j}{L^{3+|j|}}$ for large $|j|$
- Can we do more?
- What about $\bar{c}_0(\mathbf{v}_\ell)$? Magic velocity $\bar{c}_0(\mathbf{v}_\ell^{\mathrm{magic}}) = 0$



- Lattice perturbation theory
- Scalar QED (pointlike)
- vegas: $\Delta = \sum_{\mathbf{k}\neq\mathbf{0}} - \int_{\mathbf{k}}$
- Captures exponentials

$$\frac{\Delta^{\mathrm{QED_L}} m_P^2(L) - \Delta^{\mathrm{QED_r}} m_P^2(L)}{m_{P,0}^2} \overset{\text{point}}{=} \frac{-1}{4\pi^2(m_P L)} + \frac{-1}{2\pi(m_P L)^2} + \mathcal{O}[e^{-m_P L}]$$



RBC-UKQCD 23:

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- Could eliminate our major systematic error with $\mathrm{QED_r}$
- New RBC-UKQCD calculation with $\mathrm{QED_r}$
- Address volume and sub-leading uncertainties

Conclusions and outlook

- Searching for/constraining new physics with precision calculations
- Systematic finite-volume effects crucial
- Leptonic decays: Solved bottleneck, $\mathrm{QED_r}$
- $\mathrm{QED_r}$: General, also other processes
- Semi-leptonic decays, $\pi\pi$ scattering, ...

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