High order ChPT and the Lattice

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Vetenskapsrådet

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Introduction

- Had other obligations the previous three times (Stavanger 2023 , Helsinki 2022, Odense 2021)
- No direct lattice work in Sweden (1980s-1990s Carsten Petersson, Anders Irbäck Lund)
- Lots of lattice related work
- My two most recent PhD students: Nils Hermansson Truedsson and Mattias Sjö now (mainly) in lattice
- Many contacts with the lattice since long back (1985...)
- First plan: talk about 3 pion scattering and the lattice → Mattias not enough real lattice for him yet to talk about
- Give an intro to some of the places ChPT is useful for
- `bijnens@thep.lu.se` → `johan.bijnens@thep.lu.se` → `johan.bijnens@hep.lu.se` → `johan.bijnens@fysik.lu.se`

Introduction

- Some useful links
  - https://home.thep.lu.se/~bijnens/chpt/
    List of introductory references and lots of formulas
  - https://home.thep.lu.se/~bijnens/chiron/
    C++ set of programs and classes (local)
  - https://github.com/johanbijnens/CHIRON
    Same but access to more recent stuff
- School in Bad Honnef 2023: Methods of Effective Field Theory and Lattice Field Theory
  - https://indico.ph.tum.de/event/7405/
  - My lectures: https://indico.ph.tum.de/event/7405/contributions/7774/attachments/5305/6847/badhonnef23full.pdf

Resources: partial list

- Books
  - Ulf-G Meißner and Akaki Rusetsky, Effective Field Theories, Cambridge University press 2022, DOI: 10.1017/9781108689038
  - Alexey A Petrov and Andrew E Blechman, Effective Field Theories, World Scientific 2016, DOI: 10.1142/8619
  - Stefan Scherer and Matthias R. Schindler, Matthias, A Primer for Chiral Perturbation Theory, Springer 2012, DOI: 10.1007/978-3-642-19254-8
- Earlier lectures in schools
  - Les Houches summer school: EFT in Particle Physics and Cosmology, 2017 DOI 10.1093/oso/9780198855743.001.0001
  - Antonio Pich, Effective Field Theory with Nambu-Goldstone Modes, arxiv:1804.05664
  - Maarten Golterman, Applications of chiral perturbation theory to lattice QCD, arxiv:0912.4042
  - Stephen Sharpe, Applications of Chiral Perturbation theory to lattice QCD hep-lat/0607016
- FLAG report: Y. Aoki *et al.*, Eur. Phys. J. C **82** (2022) 869 arXiv:2111.09849

Chiral symmetry in QCD

Chiral Symmetry

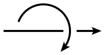
QCD: 3 light quarks: equal mass: interchange: $U(3)_V$

But $$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_{q=u,d,s} \left[i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q \left(\bar{q}_R q_L + \bar{q}_L q_R \right) \right]$$

So if $m_q = 0$ then $U(3)_L \times U(3)_R$.

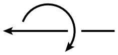
$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \to U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \text{ and } \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \to U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

Can also see that left right independent via



$v < c,\ m_q \neq 0 \Longrightarrow$

$v = c,\ m_q = 0 \not\Longrightarrow$



Chiral symmetry is broken: experiment

Chiral Symmetry

QCD: 3 light quarks: equal mass: interchange: $U(3)_V$

But $$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_{q=u,d,s} \left[i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q \left(\bar{q}_R q_L + \bar{q}_L q_R \right) \right]$$

- Hadrons do not come in parity doublets: symmetry must be broken
- A few very light hadrons: $\pi^0 \pi^+ \pi^-$ and also K, η
- Both can be understood from spontaneous Chiral Symmetry Breaking (SCSB)
- $U(3)_L \times U(3)_R = SU(3)_L \times SU(3)_L \times U(1)_V \times U(1)_A$
- $U(1)_V$ is baryon number (no effect for mesons)
- Anomaly: breaks $U(1)_A$ so η' is heavy and not a (pseudo-)Goldstone boson

Chiral symmetry is broken: theory

- Chiral symmetry in QCD is spontaneously broken by $\langle \bar{u}u + \bar{d}d + \bar{s}s \rangle$ and $\bar{q}q = \bar{q}_L q_R + \bar{q}_R q_L$
- Not invariant under $q_L \to g_L q_L$, $q_R \to g_R q_R$ with $(g_L, g_R) \in G = SU(3)_L \times SU(3)_R$
- But invariant if $g_L = g_R$: H is $SU(3)_V$
- This is the symmetry we will use: $SU(3)_L \times SU(3)_R / SU(3)_V$
- $\langle \bar{q}q \rangle$ is non-zero on the lattice
- There are 8 candidates for Goldstone boson (lighter than other hadrons)
- Explains $m_\eta^2 = (4m_K^2 - m_\pi^2)/3$ rather than $m_\eta = (4m_K - m_\pi)/3$
- Vector global symmetries in a vector gauge theory remain unbroken (Vafa-Witten)
- True in the large N_c limit (Coleman-Witten)
- Assuming confinement anomalies cannot be matched without it ('t Hooft)
- All of PCAC (and ChPT) phenomenology

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Goldstone bosons

- $SU(3)_L \times SU(3)_R / SU(3)_V \approx SU(3)$
- This parametrizes the possible vacua;
  waves in vacuum direction=Goldstone bosons
- This particular form simplifies dealing with the coset G/H
- Reason: broken generators do not form a group but parametrize a unitary matrix
- $U = \exp\left(\frac{i\sqrt{2}\Phi}{F}\right) \qquad \Phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}$
- Red: two-flavour, add black: three-flavour ChPT
- $U \to g_R U g_L^\dagger$ under $(g_L, g_R) \in SU(3)_L \times SU(3)_R$
- Φ transforms linearly under $h = g_L = g_R$, nonlinear under G/H.

Some extensions

- Can add more effects via external fields/spurions
  - example quark mass term; add external field $\widetilde{\chi} \to g_R \widetilde{\chi} g_L^\dagger$
  - $-\overline{q}_R \widetilde{\chi} q_L + \mathrm{h.c.}$ is chirally invariant
  - quark mass term reproduced by $\langle \widetilde{\chi} \rangle = \mathrm{diag}(m_u, m_d, m_s)$
- Make chiral symmetry local by adding external vector fields
  - $l_\mu \to g_L l_\mu g_L^\dagger - i g_L \partial_\mu g_L^\dagger$
  - $(\partial_\mu - i g_S G_\mu) q_L \to (\partial_\mu - i g_S G_\mu - i l_\mu) q_L)$
  - Same for r_μ and $g_R, \ldots$
  - Ward identities automatic
- Lowest order ChPT Lagrangian

$$\mathcal{L}_2 = \frac{F^2}{4} \left\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U \chi^\dagger \right\rangle \qquad D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu,\ \chi = 2B\widetilde{\chi}$$

Higher order Lagrangians

- Can construct objects that transform simply under $h = g_L = g_R$ case and then go over to a full chiral transformation (CCWZ)
- Many variations on notations
- Can add many more effects:
-   - Look at quark operator
    - Add a new spurion
    - Construct Chiral Lagrangians with tghe extra spurion
- Many Lagrangians known to very high order but leads to many free parameters

	N_f		$N_f = 3$		$N_f = 2$		
	Total	Contact	Total	Contact	Total	Contact	
p^2	2	0	2	0	2	0	Weinberg 1967
p^4	13	2	12	2	10	3	Gasser, Leutwyler 1984,1985
p^6	115	3	94	4	56	4	JB, Colangelo, Ecker, 1999
p^8	1862	22	1254	21	475	23	JB, NHT, Wang 2018

Other effects

- Many lattice effects can be copied onto ChPT
- Partially quenched
- Lattice improvements (use the Symanzik improvement operators as extra quark operators)
- Staggered (different underlying dof)
- Finite volume and twisted boundary conditions
- Twisted mass
- Weak decays
- Electromagnetic effects
- baryons
- ...

Two flavour ChPT: mass and decay constant

- Lowest order: Gell-Mann, Oakes, Renner (1968)
- Chiral logarithm Langacker, Pagels (1973)
- Full NLO (and properly starting ChPT) Gasser-Leutwyler (1984)
- NNLO Buergi (1996), JB, Colangelo, Ecker, Gasser, Sainio (1996)
- NNNLO JB, Hermansson-Truedsson (1710.01901)

Methods used

- LO and Chiral logs: current algebra
- NLO: Feynman diagrams (by hand) and direct expansion of functional integral (with REDUCE)
- NNLO: Feynman diagrams (with a little help from FORM)
- NNLO: Feynman diagrams purely with FORM
- Main stumbling block: integrals
  - Reduction to master integrals with REDUZE Studerus (2009)
  - Master Integrals known
    Laporta-Remiddi (1996); Melnikov, van Ritbergen (2001)
- Lots of book-keeping: FORM
- Checks:
  - All nonlocal divergences must cancel
  - Use different parametrizations of the Lagrangian
  - Agree with known leading log result JB, Carloni (2009)

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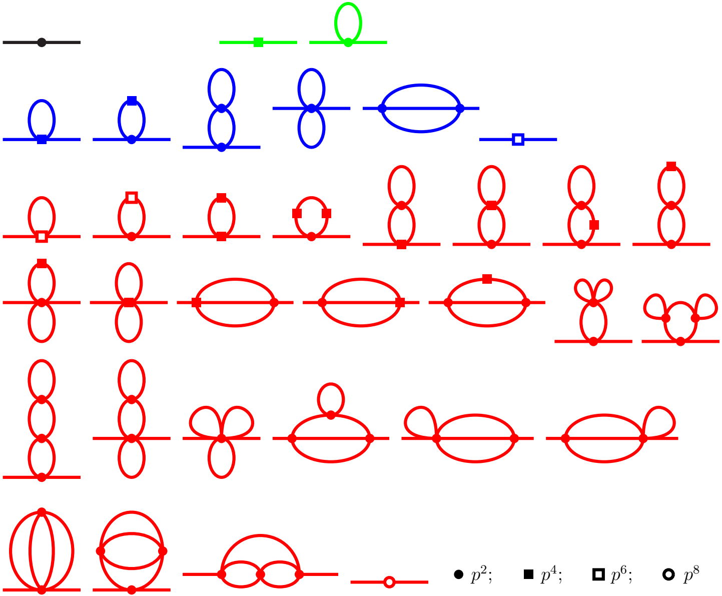
Diagrams



ChPT and the lattice

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Results: LO or x-expansion/physical or ξ-expansion

- $x = \dfrac{M^2}{16\pi^2 F^2}, \qquad L_x = \log \dfrac{M^2}{\mu^2}, \qquad M^2 = 2B\hat{m}$

$$\frac{M_\pi^2}{M^2} = 1 + x\left(a_{11}^M L_x + a_{10}^M\right) + x^2\left(a_{22}^M L_x^2 + a_{21}^M L_x + a_{20}^M\right) + x^3\left(a_{33}^M L_x^3 + a_{32}^M L_x^2 + a_{31}^M L_x + a_{30}^M\right) + \cdots$$

$$\frac{F_\pi}{F} = 1 + x\left(a_{11}^F L_x + a_{10}^F\right) + x^2\left(a_{22}^F L_x^2 + a_{21}^F L_x + a_{20}^F\right) + x^3\left(a_{33}^F L_x^3 + a_{32}^F L_x^2 + a_{31}^F L_x + a_{30}^F\right) + \cdots$$

- $\xi = \dfrac{M_\pi^2}{16\pi^2 F_\pi^2}, \qquad L_\pi = \log \dfrac{M_\pi^2}{\mu^2}$
- $$\frac{M^2}{M_\pi^2} = 1 + \xi\left(b_{11}^M L_\pi + b_{10}^M\right) + \xi^2\left(b_{22}^M L_\pi^2 + b_{21}^M L_\pi + b_{20}^M\right) + \xi^3\left(b_{33}^M L_\pi^3 + b_{32}^M L_\pi^2 + b_{31}^M L_\pi + b_{30}^M\right) + \cdots$$
- $$\frac{F}{F_\pi} = 1 + \xi\left(b_{11}^F L_\pi + b_{10}^F\right) + \xi^2\left(b_{22}^F L_\pi^2 + b_{21}^F L_\pi + b_{20}^F\right) + \xi^3\left(b_{33}^F L_\pi^3 + b_{32}^F L_\pi^2 + b_{31}^F L_\pi + b_{30}^F\right) + \cdots$$

Results $\tilde{l}_i = 16\pi^2 l_i^r$, $\tilde{c}_i = (16\pi^2)^2 c_i^r$

a_{11}^M	$\frac{1}{2}$
a_{10}^M	$2\tilde{l}_3$
a_{22}^M	$\frac{17}{8}$
a_{21}^M	$-3\tilde{l}_3 - 8\tilde{l}_2 - 14\tilde{l}_1 - \frac{49}{12}$
a_{20}^M	$64\tilde{c}_{18} + 32\tilde{c}_{17} + 96\tilde{c}_{11} + 48\tilde{c}_{10} - 16\tilde{c}_9 - 32\tilde{c}_8 - 16\tilde{c}_7$ $-32\tilde{c}_6 + \tilde{l}_3 + 2\tilde{l}_2 + \tilde{l}_1 + \frac{193}{96}$
a_{33}^M	$\frac{103}{24}$
a_{32}^M	$\frac{23}{2}\tilde{l}_3 - 11\tilde{l}_2 - 38\tilde{l}_1 - \frac{91}{24}$
a_{31}^M	$-416\tilde{c}_{18} - 208\tilde{c}_{17} - 32\tilde{c}_{16} + 96\tilde{c}_{14} + 8\tilde{c}_{13} - 48\tilde{c}_{12}$ $-384\tilde{c}_{11} - 192\tilde{c}_{10} + 72\tilde{c}_9 + 144\tilde{c}_8 + 72\tilde{c}_7 + 64\tilde{c}_6 - 8\tilde{c}_5$ $-56\tilde{c}_4 + 16\tilde{c}_3 + 32\tilde{c}_2 - 96\tilde{c}_1 - 8\tilde{l}_3^2 - 48\tilde{l}_3\tilde{l}_2 - 84\tilde{l}_3\tilde{l}_1$ $-\frac{88}{3}\tilde{l}_3 - \frac{231}{10}\tilde{l}_2 - \frac{69}{5}\tilde{l}_1 - \frac{74971}{8640}$
a_{30}^M	contains free p^8 LECs (and a lot more terms)

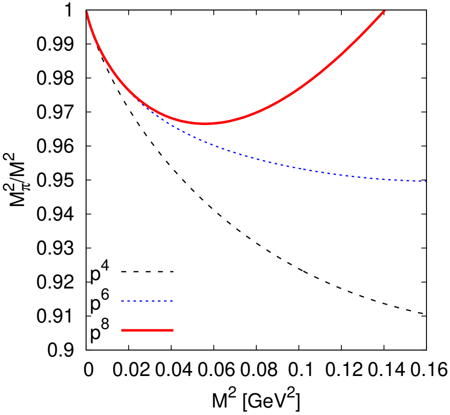
Results: comments

- Similar tables for a_i^F, b_i^M, b_i^F
- Coefficients depend on scale μ, but whole expression is μ-independent
- Can be rewritten in terms of scales in the logarithm rather than in terms of LECs à la FLAG
- Leading log: a number
- NLL: depends on l_i^r
- NNLL: depends on c_i^r
- For the mass all needed c_i^r can be had from mass, decay-constant and $\pi\pi$ parameters fitted to two-loop or p^6 (i.e. $r_M, r_F, r_1, \ldots, r_6$).
- For decay need one more

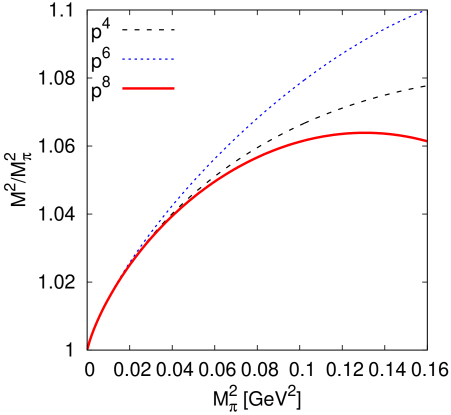
Pion mass

$F_\pi = 92.2$ MeV, $F = F_\pi/1.037$, $\bar{l}_1 = -0.4$, $\bar{l}_2 = 4.3$, $\bar{l}_3 = 3.41$, $\bar{l}_4 = 4.51$,

r_i from JB et al 1997, other $c_i^r = 0$, $\mu = 0.77$ GeV



x-expansion (F-fixed)



ξ-expansion (F_π fixed)

ξ-expansion converges notably better

Fit to lattice

- Try to fit to lattice data from BMW 1310.3626
- They did a two-loop fit
- Three-loop fit done in bachelor thesis Alexandra Wernersson (2020) `https://lup.lub.lu.se/student-papers/search/publication/8986120`
- 57 data points (47 below 500 MeV)
- Try both x and ξ-expansions

Results

Order	χ^2	χ^2 with p^4	χ^2 with p^4+p^6
NLO	161.2	137.6	120.8
NNLO (Λ_1, Λ_2 free)	55.8	50.9	50.9
NNLO (Λ_{12} free)	55.8	50.9	50.8
NNNLO (Λ_1, Λ_2 free)	52.3	48.5	48.9
NNNLO (Λ_{12} free)	53.0	48.8	49.5
NNNLO (Λ_{12} default)	48.6	48.6	49.5

Order	χ^2	χ^2 with p^4	χ^2 with p^4+p^6
NLO	104.8	88.3	81.9
NNLO ($\bar\ell_1, \bar\ell_2$ free)	62.6	56.0	55.2
NNLO ($\bar\ell_{12}$ free)	62.6	57.7	57.4
NNNLO ($\bar\ell_1, \bar\ell_2$ free)	56.8	51.5	51.9
NNNLO ($\bar\ell_{12}$ free)	55.2	51.0	51.9
NNNLO ($\bar\ell_{12}$ default)	51.8	51.0	51.8

Columns: No FV, FV p^4, FV p^6

Rows: $\pi\pi$-scattering LECs

x expansion

ξ expansion

Conclusions of the fit

- ξ-fit much better at NLO
- Difference between the two much less at higher orders
- ChPT finite volume corrections improve the fit
- NNNLO not really visible in the fits

Three flavour

- There are more lattice groups that have done NNLO fits to LECs (RBC-UKQCD,MILC)
- Three flavour fits exists as well
- Try here: Bachelor thesis Hector Tiblom (20221) `https://lup.lub.lu.se/student-papers/search/publication/9051852`
- Conclusion: difficult to fit properly, fits go off due to small range in m_s
- Really need to cooperate with lattice groups to do this (correlations)
- Reason for CHIRON: make the long programs accessible
- One still sees many NLO fits only

Finite volume

- J. Gasser and H. Leutwyler, "Spontaneously Broken Symmetries: Effective Lagrangians at Finite Volume," Nucl. Phys. B **307** (1988), 763-778
- Lattice calculations done in finite volume
- These effect go as $\exp(-m_\pi L)$
- For high precision need to be corrected for
- Known two two loops for masses, decay constants and vacuum expectation values (two and three flavours)
- JB, Ghorbani 2006, Colangelo, Haefeli 2006, JB, Rössler 2015
- Integrals JB, Boström, Lähde, 2013

The underlying formulas

- Underlying formula in one dimension
  periodic boundary condition $F(x=0) = F(x=L)$

$$\int \frac{dp}{2\pi} F(p) \longrightarrow \frac{1}{L} \sum_{p_n = 2\pi n/L} F(p_n) \equiv \int_L \frac{dp}{2\pi} F(p)$$

- Poisson summation formula

$$\frac{1}{L} \sum_{p_n = 2\pi n/L} F(p_n) = \sum_{\ell = nL} \int \frac{dp}{2\pi} e^{i\ell p} F(p)$$

- If twist angle θ, $\phi(L) = e^{-i\theta}\phi(0)$: $p_n = \frac{2\pi}{L} n + \frac{\theta}{L}$
- Poisson summation formula

$$\frac{1}{L} \sum_{p_n = 2\pi n/L + \theta/L} F(p_n) = \sum_{\ell = nL} \int \frac{dp}{2\pi} e^{i(\ell p - \ell(\theta/L))} F(p)$$

One-loop tadpole

$$\langle X \rangle = \int_V \frac{d^d r}{(2\pi)^d} \frac{X}{(r^2+m^2)^n},$$

Poisson trick for three spatial dimensions:

$$\langle X \rangle = \sum_{l_r} \int \frac{d^d r}{(2\pi)^d} \frac{X\, e^{il_r\cdot r - il_r\cdot\Theta}}{(r^2+m^2)^n},$$

$l_r = (0, n_1 L, n_2 L, n_3 L)$, $\Theta = (0, \vec{\theta}/L)$
Split in infinite volume $l_r = 0$ term and rest

$$\langle X \rangle = \langle X \rangle^\infty + \langle X \rangle^V$$

Bring up denominator using 'α' or Schwinger parametrization: $1/a = \int_0^\infty d\lambda e^{-\lambda a}$

$$\langle 1 \rangle^V = \frac{1}{\Gamma(n)} \sum_{l_r}{}' \int \frac{d^d r}{(2\pi)^d} \int_0^\infty d\lambda \lambda^{n-1}\, e^{il_r\cdot r - il_r\cdot\Theta} e^{-\lambda(r^2+m^2)}$$

$\sum_{l_r}'$ means sum without $l_r = 0$ (all components zero)

One-loop tadpole

Shift $r = \bar{r} + i l_r/(2\lambda)$ to get

$$\langle 1 \rangle^V = \frac{1}{\Gamma(n)} {\sum_{l_r}}' \int_0^\infty d\lambda \lambda^{n-1}\, e^{-\lambda m^2 - \frac{l_r^2}{4\lambda} - i l_r \cdot \Theta} \int \frac{d^d \bar{r}}{(2\pi)^d} e^{-\lambda \bar{r}^2}$$

Master formula for tadpoles:

$$\boxed{\langle 1 \rangle^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} {\sum_{l_r}}' \int_0^\infty d\lambda \lambda^{n-\frac{d}{2}-1}\, e^{-\lambda m^2 - \frac{l_r^2}{4\lambda} - i l_r \cdot \Theta}}$$

- Do the λ integral — Gasser, Leutwyler, 1988
  leads to sums over Bessel functions
- Do the $\sum_{l_r}$ — Becirevic, Villadoro, 2003
  leads to an integral over Jacobi theta functions

One-loop tadpole: Bessel functions

We use $\mathcal{K}_\nu(Y,Z) = \int_0^\infty d\lambda\,\lambda^{\nu-1} e^{-Z\lambda - Y/\lambda} = 2\left(\frac{Y}{Z}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{YZ}\right)$ to obtain

$$\langle 1\rangle^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} \sum_{l_r}{}' e^{-il_r\cdot\Theta} \mathcal{K}_{n-\frac{d}{2}-1}\left(m^2, \frac{l_r^2}{4}\right)$$

- $d = 4 - 2\epsilon$ and can expand in ϵ
- Triple sum can be simplified: $\sum_{l_r}' f(l_r^2) = \sum_{k>0} x(k) f(k)$

  $k = l_r^2$ and $x(k)$ number of times $l_r^2 = kL^2$
- $K_i(mL\sqrt{k}) \approx \sqrt{\frac{\pi}{2mL\sqrt{k}}} e^{-mL\sqrt{k}}$
- The sums are referred to as "going around the world"

One-loop tadpole: (Jacobi) theta functions

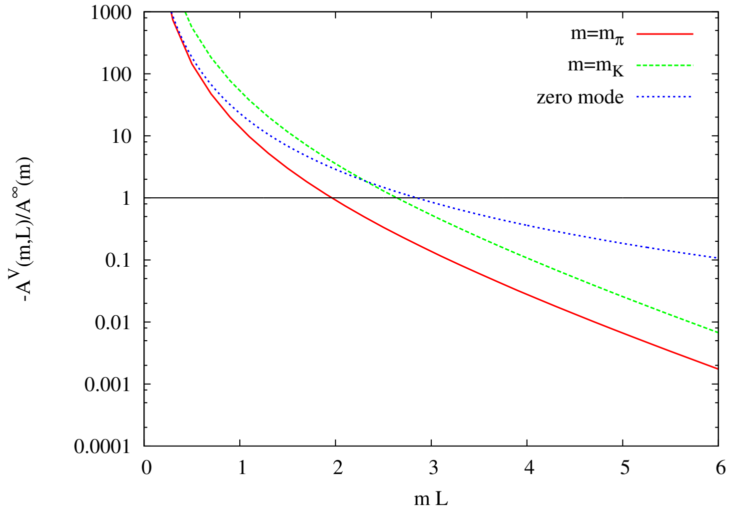
Define $\theta_3(u|\tau) = \sum_n e^{i\pi\tau n^2 + 2\pi i u n}$ which satisfies

$\theta_3(u+n|\tau) = \theta_3(u|\tau)$ and $\theta_3(u|\tau) = \frac{1}{\sqrt{-i\tau}} e^{-\pi i \frac{u^2}{\tau}} \theta_3\left(\frac{u}{\tau}\middle|\frac{-1}{\tau}\right)$

$$\langle 1 \rangle^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} \int_0^\infty d\lambda \lambda^{n-\frac{d}{2}-1}\, e^{-\lambda m^2} \times \left[\Pi_{j=x,y,z}\theta_3\left(-\theta_j/(2\pi)|iL^2/(4\pi\lambda)\right) - 1\right]$$

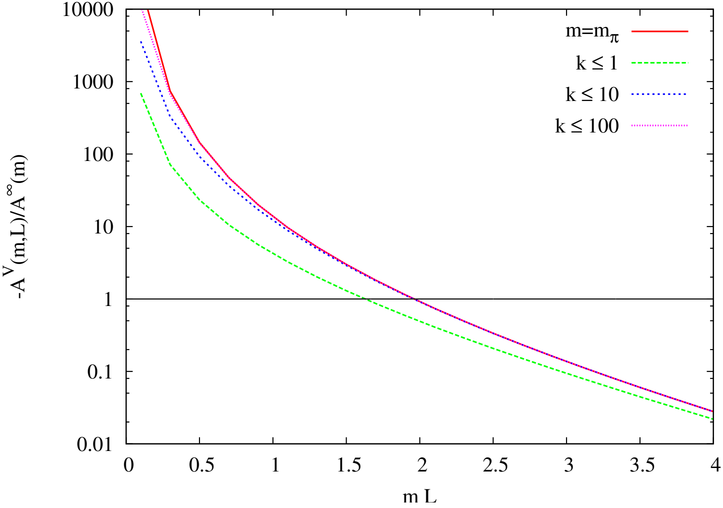
- No twisting, goes to a cubed theta function
- rescale λ to get L out of the argument
- Small λ, use the identity above, large λ sum converges fast

A numerical example



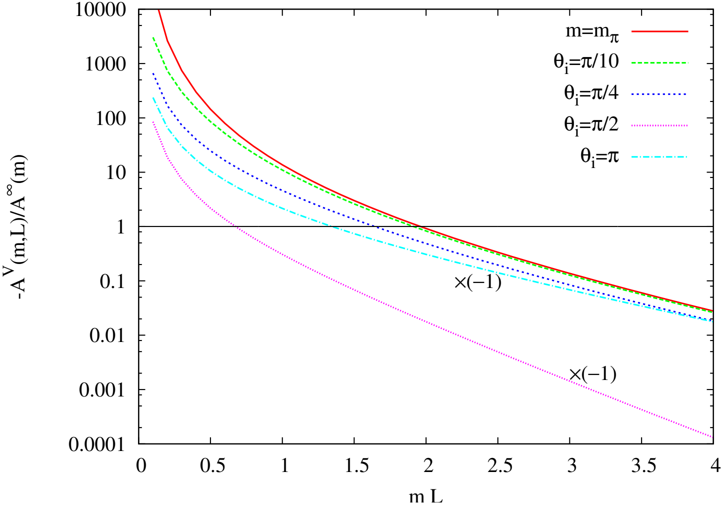
Relative correction to the infinite volume integral $A = \int_V \frac{d^d p}{(2\pi)^d} \frac{1}{p^2+m^2}$

A numerical example



Convergence of the Bessel sum as a function of k

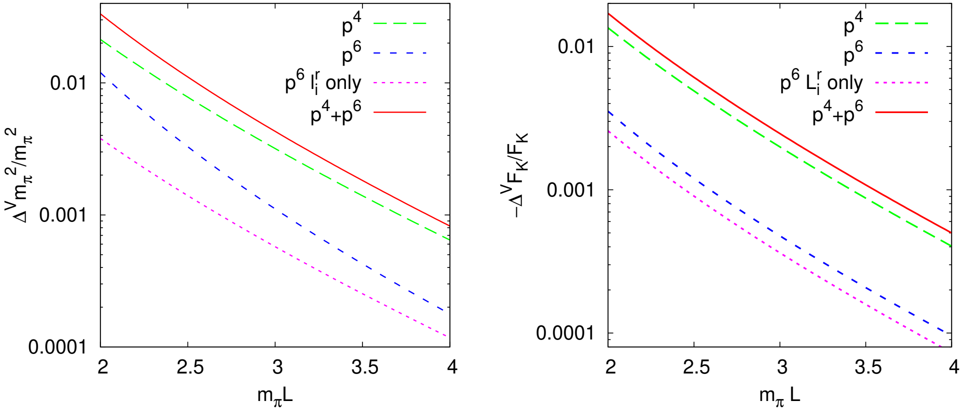
A numerical example



For various valus of $\theta_x = \theta_y = \theta_z = \theta_i$

For $\theta = \pi/2$ all terms with an $(l_r)_i$ odd cancel

Some results at two loops



JB, Rössler, 2015

Some caveats for finite volume

- The volume breaks rotational symmetry to the cubic group
- It also breaks Lorentz (boost invariance)
- Consequences:
  - The 'mass' depends on the momentum, i.e. the dispersion relation is not simply $E^2 = \vec{p}^2 + m^2$
  - There are more form-factors and they depend on E and $\vec{p}$ beyond $p^2 = E^2 + \vec{p}^2$
  - Example $\Pi^V_{\mu\nu}\nu \left(q_\mu q_\nu - g_{\mu\nu} q^2\right) \Pi^V(q^2)$

Partial quenching

- Fermions in lattice QCD: integrated out explicitly
- $\int [dqd\bar{q}][dG]\bar{q}(0)q(y)\exp\left\{i\int d^4x\left(i\bar{q}(q)i\not{D}q(x)-\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}\right)\right\}=$
  $\int [dG]\mathrm{Prop}(0,y)\det\not{D}\exp\left\{i\int d^4x\left(-\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}\right)\right\}$
- Partial quenching:
  $$\int [dG]\underbrace{\mathrm{Prop}(0,y)}_{\text{Valence quarks}}\quad\underbrace{\det\not{D}}_{\text{sea quarks}}\exp\left\{i\int d^4x\left(-\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}\right)\right\}$$
- In real QCD: same masses, but can be made different in lattice QCD
- Advantage: easier to change only valence masses (or twist them)
- Quenched: drop $\det\not{D}$
- Partially quenched QCD is continually connected to QCD

Partial quenching

- Fermions in lattice QCD: integrated out explicitly
- $\int [dqd\bar{q}][dG]\bar{q}(0)q(y) \exp\left\{ i \int d^4x \left(i\bar{q}(q) i\not{D} q(x) - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \right) \right\} =$
  $\int [dG]\mathrm{Prop}(0,y) \det \not{D} \exp\left\{ i \int d^4x \left(-\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \right) \right\}$
- Partial quenching:
  $$\int [dG] \underbrace{\mathrm{Prop}(0,y)}_{\text{Valence quarks}} \quad \underbrace{\det \not{D}}_{\text{sea quarks}} \exp\left\{ i \int d^4x \left(-\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \right) \right\}$$
- In real QCD: same masses, but can be made different in lattice QCD
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Partially quenched ChPT

- Quenched logarithms: Morel 1987, Sharpe 1990, 1992
  Method: following quark lines explicitly through diagrams
- Using "ghost quarks": Bernard, Golterman 1992, 1993
- Note it is not a field theory but lots of the diagrammatic proofs go through Bernard, Golterman, 2013
- Best method and introduction: S. R. Sharpe and N. Shoresh, Phys. Rev. D **64** (2001), 114510 [arXiv:hep-lat/0108003 [hep-lat]]
- Lagrangians are those of N_F ChPT (replica trick Damgaard 2000)

Partially Quenched

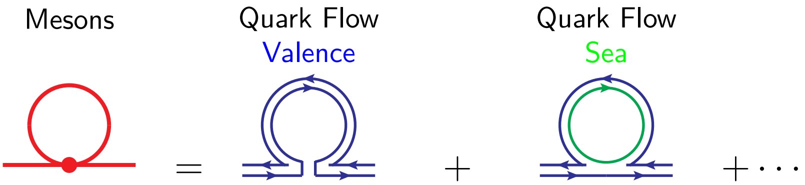
- Why do this?
  - Is not Quenched: Real QCD is continuous limit from Partially Quenched
  - More handles to turn:
    - Allows more systematic studies by varying parameters
    - Sometimes allows to disentangle things from different observables
- Why not do this
  - It is not QCD as soon as Valence$\neq$Sea
  - not a Quantum Field Theory
  - No unitarity
  - No clusterdecomposition
  - No CPT theorem
  - No spin statistics theorem

Partially Quenched

Do anyway

- Cheap computationally
- Allows extra studies
- Hope it works near real QCD case
- Is a well defined Euclidean statistical model

ChPT and Lattice QCD

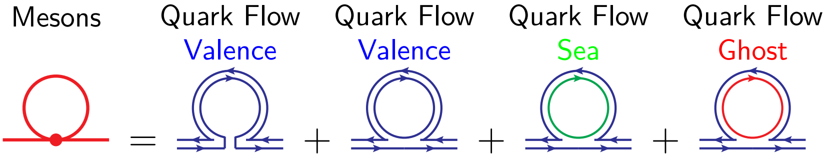


Valence is easy to deal with in lattice QCD
Sea is very difficult

They can be treated separately: i.e. different quark masses
Partially Quenched ChPT (PQChPT)

PQChPT at Two Loops: General

Add ghost quarks: remove the unwanted free valence loops



Possible problem: QCD $\Longrightarrow$ ChPT relies heavily on unitarity

Partially quenched: at least one dynamical sea quark

$\Longrightarrow$ Φ_0 is heavy: remove from PQChPT

Symmetry group becomes $SU(n_v + n_s|n_v) \times SU(n_v + n_s|n_v)$ (approximately)

PQChPT at Two Loops: General

Essentially all manipulations from ChPT go through to PQChPT when changing trace to supertrace and adding fermionic variables

Exceptions: baryons and Cayley-Hamilton relations

So Luckily: can use the n flavour work in ChPT at two loop order to obtain for PQChPT: Lagrangians and infinities

Very important note: ChPT is a limit of PQChPT
$\Longrightarrow$ LECs from ChPT are linear combinations of LECs of PQChPT with the same number of sea quarks.

E.g. $L_1^r = L_0^{r(3pq)}/2 + L_1^{r(3pq)}$

PQChPT at Two Loop

valence equal mass, 3 sea equal mass:

$m^2_{\pi^+}$: JB,Danielsson,Lähde, hep-lat/0406017

Other mass combinations:

F_{π^+}: JB,Lähde, hep-lat/0501014

F_{π^+}, $m^2_{\pi^+}$ two sea quarks: JB,Lähde, hep-lat/0506004

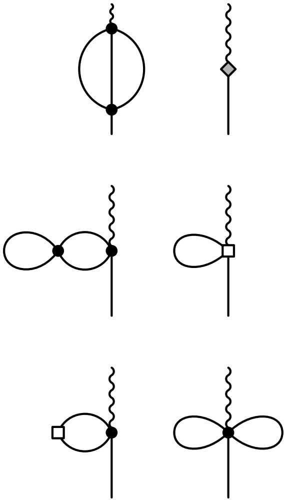
$m^2_{\pi^+}$: JB,Danielsson,Lähde, hep-lat/0602003

Neutral masses: JB,Danielsson, hep-lat/0606017

Actual Calculations:
- heavy use of FORM Vermaseren
- use PQ without super Φ_0 in supersymmetric formalism
- Main problem: sheer size of the expressions

Iso breaking from lattice data: a and L extrapolations needed

Long Expressions



$\Longrightarrow$

$$
\begin{aligned}
\delta^{(6)22}_{\text{loops}} = \;& \pi_{16}\, L^r_0 \left[4/9\,\chi_\eta\chi_4 - 1/2\,\chi_1\chi_3 + \chi_{13}^2 - 13/3\,\bar\chi_1\chi_{13} - 35/18\,\bar\chi_2\right] - 2\,\pi_{16}\, L^r_1\, \chi_{13}^2 \\
& - \pi_{16}\, L^r_2 \left[11/3\,\chi_\eta\chi_4 + \chi_{13}^2 + 13/3\,\bar\chi_2\right] + \pi_{16}\, L^r_3 \left[4/9\,\chi_\eta\chi_4 - 7/12\,\chi_1\chi_3 + 11/6\,\chi_{13}^2 - 17/6\,\bar\chi_1\chi_{13} - 43/36\,\bar\chi_2\right] \\
& + \pi_{16}^2 \left[-15/64\,\chi_\eta\chi_4 - 59/384\,\chi_1\chi_3 + 65/384\,\chi_{13}^2 - 1/2\,\bar\chi_1\chi_{13} - 43/128\,\bar\chi_2\right] - 48\, L^r_4 L^r_5\, \bar\chi_1\chi_{13} - 72\, L^{r2}_4\, \bar\chi_1^2 \\
& - 8\, L^{r2}_5\, \chi_{13}^2 + \bar A(\chi_p)\,\pi_{16} \left[-1/24\,\chi_p + 1/48\,\bar\chi_1 - 1/8\,\bar\chi_1\, R^p_{q\eta} + 1/16\,\bar\chi_1\, R^c_p - 1/48\, R^p_{q\eta}\,\chi_p - 1/16\, R^p_{q\eta}\,\chi_q\right. \\
& \left. + 1/48\, R^\eta_{pp}\,\chi_\eta + 1/16\, R^c_p\,\chi_{13}\right] + \bar A(\chi_p)\, L^r_0 \left[8/3\, R^p_{q\eta}\,\chi_p + 2/3\, R^c_p\,\chi_p + 2/3\, R^d_p\right] + \bar A(\chi_p)\, L^r_3 \left[2/3\, R^p_{q\eta}\,\chi_p\right. \\
& \left. + 5/3\, R^c_p\,\chi_p + 5/3\, R^d_p\right] + \bar A(\chi_p)\, L^r_4 \left[-2\,\bar\chi_1\bar\chi^{pp}_{\eta\eta 0} - 2\,\bar\chi_1\, R^p_{q\eta} + 3\,\bar\chi_1\, R^c_p\right] + \bar A(\chi_p)\, L^r_5 \left[-2/3\,\bar\chi^{pp}_{\eta\eta 1} - R^p_{q\eta}\,\chi_p\right. \\
& \left. + 1/3\, R^p_{q\eta}\,\chi_q + 1/2\, R^c_p\,\chi_p - 1/6\, R^c_p\,\chi_q\right] + \bar A(\chi_p)^2 \left[1/16 + 1/72\,(R^p_{q\eta})^2 - 1/72\, R^p_{q\eta} R^c_p + 1/288\,(R^c_p)^2\right] \\
& + \bar A(\chi_p)\bar A(\chi_{ps}) \left[-1/36\, R^p_{q\eta} - 5/72\, R^p_{s\eta} + 7/144\, R^c_p\right] - \bar A(\chi_p)\bar A(\chi_{qs}) \left[1/36\, R^p_{q\eta} + 1/24\, R^p_{s\eta} + 1/48\, R^c_p\right] \\
& + \bar A(\chi_p)\bar A(\chi_\eta) \left[-1/72\, R^p_{q\eta} R^v_{\eta 13} + 1/144\, R^c_p R^v_{\eta 13}\right] + 1/8\,\bar A(\chi_p)\bar A(\chi_{13}) + 1/12\,\bar A(\chi_p)\bar A(\chi_{46})\, R^\eta_{pp} \\
& + \bar A(\chi_p)\bar B(\chi_p,\chi_p;0) \left[1/4\,\chi_p - 1/18\, R^p_{q\eta} R^c_p\,\chi_p - 1/72\, R^p_{q\eta} R^d_p + 1/18\,(R^c_p)^2\,\chi_p + 1/144\, R^c_p R^d_p\right] \\
& + \bar A(\chi_p)\bar B(\chi_p,\chi_\eta;0) \left[1/18\, R^\eta_{pp} R^c_p\,\chi_p - 1/18\, R^\eta_{13} R^c_p\,\chi_p\right] + \bar A(\chi_p)\bar B(\chi_q,\chi_q;0) \left[-1/72\, R^p_{q\eta} R^d_q + 1/144\, R^c_p R^d_q\right] \\
& - 1/12\,\bar A(\chi_p)\bar B(\chi_{ps},\chi_{ps};0)\, R^p_{s\eta}\,\chi_{ps} - 1/18\,\bar A(\chi_p)\bar B(\chi_1,\chi_3;0)\, R^q_{p\eta} R^c_p\,\chi_p \\
& + 1/18\,\bar A(\chi_p)\bar C(\chi_p,\chi_p,\chi_p;0)\, R^c_p R^d_p\,\chi_p + \bar A(\chi_p;\varepsilon)\,\pi_{16} \left[1/8\,\bar\chi_1 R^p_{q\eta} - 1/16\bar\chi_1\, R^c_p - 1/16\, R^c_p\,\chi_p - 1/16\, R^d_p\right] \\
& + \bar A(\chi_{ps})\,\pi_{16} \left[1/16\,\chi_{ps} - 3/16\,\chi_{qs} - 3/16\,\bar\chi_1\right] - 2\,\bar A(\chi_{ps})\, L^r_0\,\chi_{ps} - 5\,\bar A(\chi_{ps})\, L^r_3\,\chi_{ps} - 3\,\bar A(\chi_{ps})\, L^r_4\,\bar\chi_1 \\
& + \bar A(\chi_{ps})\, L^r_5\,\chi_{13} + \bar A(\chi_{ps})\bar A(\chi_\eta) \left[7/144\, R^\eta_{pp} - 5/72\, R^\eta_{ps} - 1/48\, R^\eta_{qq} + 5/72\, R^\eta_{qs} - 1/36\, R^\eta_{13}\right] \\
& + \bar A(\chi_{ps})\bar B(\chi_p,\chi_p;0) \left[1/24\, R^p_{s\eta}\,\chi_p - 5/24\, R^p_{s\eta}\,\chi_{ps}\right] + \bar A(\chi_{ps})\bar B(\chi_p,\chi_\eta;0) \left[-1/18\, R^\eta_{ps} R^z_{qp\eta}\,\chi_p\right. \\
& \left. - 1/9\, R^\eta_{ps} R^z_{qp\eta}\,\chi_{ps}\right] - 1/48\,\bar A(\chi_{ps})\bar B(\chi_q,\chi_q;0)\, R^d_q + 1/18\,\bar A(\chi_{ps})\bar B(\chi_1,\chi_3;0)\, R^q_{s\eta}\,\chi_s \\
& + 1/9\,\bar A(\chi_{ps})\bar B(\chi_1,\chi_3;0,k)\, R^q_{s\eta} + 3/16\,\bar A(\chi_{ps};\varepsilon)\,\pi_{16} \left[\chi_s + \bar\chi_1\right] - 1/8\,\bar A(\chi_{p4})^2 - 1/8\,\bar A(\chi_{p4})\bar A(\chi_{p6}) \\
& + 1/8\,\bar A(\chi_{p4})\bar A(\chi_{q6}) - 1/32\,\bar A(\chi_{p6})^2 + \bar A(\chi_\eta)\,\pi_{16} \left[1/16\,\bar\chi_1\, R^v_{\eta 13} - 1/48\, R^v_{\eta 13}\,\chi_\eta + 1/16\, R^v_{\eta 13}\,\chi_{13}\right] \\
& + \bar A(\chi_\eta)\, L^r_0 \left[4 R^\eta_{13}\,\chi_\eta + 2/3\, R^v_{\eta 13}\,\chi_\eta\right] - 8\,\bar A(\chi_\eta)\, L^r_1\,\chi_\eta - 2\,\bar A(\chi_\eta)\, L^r_2\,\chi_\eta + \bar A(\chi_\eta)\, L^r_3 \left[4 R^\eta_{13}\,\chi_\eta + 5/3\, R^v_{\eta 13}\chi_\eta\right] \\
& + \bar A(\chi_\eta)\, L^r_4 \left[4\,\chi_\eta + \bar\chi_1\, R^v_{\eta 13}\right] - \bar A(\chi_\eta)\, L^r_5 \left[1/6\, R^\eta_{pp}\,\chi_q + R^\eta_{13}\,\chi_{13} + 1/6\, R^v_{\eta 13}\,\chi_\eta\right] + 1/288\,\bar A(\chi_\eta)^2\,(R^v_{\eta 13})^2 \\
& + 1/12\,\bar A(\chi_\eta)\bar A(\chi_{46})\, R^v_{\eta 13} + \bar A(\chi_\eta)\bar B(\chi_p,\chi_p;0) \left[-1/36\,\bar\chi^{pp\eta}_{\eta\eta 1} - 1/18\, R^p_{q\eta} R^\eta_{pp}\,\chi_p + 1/18\, R^\eta_{pp} R^c_p\,\chi_p\right. \\
& \left. + 1/144\, R^d_p R^v_{\eta 13}\right] + \bar A(\chi_\eta)\bar B(\chi_p,\chi_\eta;0) \left[-1/18\,\bar\chi^{\eta p\eta}_{p\eta 1} + 1/18\,\bar\chi^{\eta p\eta}_{q\eta 1} + 1/18\,(R^\eta_{pp})^2 R^z_{qp\eta}\,\chi_p\right] \\
& - 1/12\,\bar A(\chi_\eta)\bar B(\chi_{ps},\chi_{ps};0)\, R^\eta_{ps}\,\chi_{ps} - \bar A(\chi_\eta)\bar B(\chi_\eta,\chi_\eta;0) \left[1/216\, R^v_{\eta 13}\,\chi_4 + 1/27\, R^v_{\eta 13}\,\chi_6\right] \\
& - 1/18\,\bar A(\chi_\eta)\bar B(\chi_1,\chi_3;0)\, R^1_{\eta\eta} R^3_{\eta\eta}\,\chi_\eta + 1/18\,\bar A(\chi_\eta)\bar C(\chi_p,\chi_p,\chi_p;0)\, R^\eta_{pp} R^d_p\,\chi_p + \bar A(\chi_\eta;\varepsilon)\,\pi_{16} \left[1/8\,\chi_\eta\right. \\
& \left. - 1/16\,\bar\chi_1\, R^v_{\eta 13} - 1/8\, R^\eta_{13}\,\chi_\eta - 1/16\, R^v_{\eta 13}\,\chi_\eta\right] + \bar A(\chi_1)\bar A(\chi_3) \left[-1/72\, R^p_{q\eta} R^c_q + 1/36\, R^1_{3\eta} R^3_{1\eta} + 1/144\, R^c_1 R^c_3\right] \\
& - 4\,\bar A(\chi_{13})\, L^r_1\,\chi_{13} - 10\,\bar A(\chi_{13})\, L^r_2\,\chi_{13} + 1/8\,\bar A(\chi_{13})^2 - 1/2\,\bar A(\chi_{13})\bar B(\chi_1,\chi_3;0,k) \\
& + 1/4\,\bar A(\chi_{13};\varepsilon)\,\pi_{16}\,\chi_{13} + 1/4\,\bar A(\chi_{14})\bar A(\chi_{34}) + 1/16\,\bar A(\chi_{16})\bar A(\chi_{36}) - 24\,\bar A(\chi_4)\, L^r_1\,\chi_4 - 6\,\bar A(\chi_4)\, L^r_2\,\chi_4 \\
& + 12\,\bar A(\chi_4)\, L^r_4\,\chi_4 + 1/12\,\bar A(\chi_4)\bar B(\chi_p,\chi_p;0)\,(R^p_{4\eta})^2\,\chi_4 + 1/6\,\bar A(\chi_4)\bar B(\chi_p,\chi_\eta;0) \left[R^p_{4\eta} R^\eta_{p4}\,\chi_4 - R^p_{4\eta} R^\eta_{q4}\,\chi_4\right] \\
& - 1/24\,\bar A(\chi_4)\bar B(\chi_\eta,\chi_\eta;0)\, R^v_{\eta 13}\,\chi_4 - 1/6\,\bar A(\chi_4)\bar B(\chi_1,\chi_3;0)\, R^1_{4\eta} R^3_{4\eta}\,\chi_4 + 3/8\,\bar A(\chi_4;\varepsilon)\,\pi_{16}\,\chi_4 \\
& - 32\,\bar A(\chi_{46})\, L^r_1\,\chi_{46} - 8\,\bar A(\chi_{46})\, L^r_2\,\chi_{46} + 16\,\bar A(\chi_{46})\, L^r_4\,\chi_{46} + \bar A(\chi_{46})\bar B(\chi_p,\chi_p;0) \left[1/9\,\chi_{46} + 1/12\, R^\eta_{pp}\,\chi_p\right. \\
& \left. + 1/36\, R^\eta_{pp}\,\chi_4 + 1/9\, R^\eta_{p4}\,\chi_6\right] + \bar A(\chi_{46})\bar B(\chi_p,\chi_\eta;0) \left[-1/18\, R^\eta_{pp}\,\chi_4 - 1/9\, R^\eta_{p4}\,\chi_6 + 1/9\, R^\eta_{q4}\,\chi_6 + 1/18\, R^\eta_{13}\,\chi_4\right] \\
& - 1/6\,\bar A(\chi_{46})\bar B(\chi_p,\chi_\eta;0,k) \left[R^\eta_{pp} - R^\eta_{13}\right] + 1/9\,\bar A(\chi_{46})\bar B(\chi_\eta,\chi_\eta;0)\, R^v_{\eta 13}\,\chi_{46} - \bar A(\chi_{46})\bar B(\chi_1,\chi_3;0) \left[2/9\,\chi_{46}\right. \\
& \left. + 1/9\, R^\eta_{p4}\,\chi_6 + 1/18\, R^\eta_{13}\,\chi_4\right] - 1/6\,\bar A(\chi_{46})\bar B(\chi_1,\chi_3;0,k)\, R^\eta_{13} + 1/2\,\bar A(\chi_{46};\varepsilon)\,\pi_{16}\,\chi_{46} \\
& + \bar B(\chi_p,\chi_p;0)\,\pi_{16} \left[1/16\,\bar\chi_1\, R^d_p + 1/96\, R^d_p\,\chi_p + 1/32\, R^d_p\,\chi_q\right] + 2/3\,\bar B(\chi_p,\chi_p;0)\, L^r_0\, R^d_p\,\chi_p \\
& + 5/3\,\bar B(\chi_p,\chi_p;0)\, L^r_3\, R^d_p\,\chi_p + \bar B(\chi_p,\chi_p;0)\, L^r_4 \left[-2\,\bar\chi_1\bar\chi^{pp}_{\eta\eta 0}\chi_p - 4\,\bar\chi_1 R^p_{q\eta}\,\chi_p + 4\,\bar\chi_1 R^c_p\,\chi_p + 3\,\bar\chi_1 R^d_p\right] \\
& + \bar B(\chi_p,\chi_p;0)\, L^r_5 \left[-2/3\bar\chi^{pp}_{\eta\eta 1}\chi_p - 4/3\, R^p_{q\eta}\,\chi_p^2 + 4/3\, R^c_p\,\chi_p^2 + 1/2\, R^d_p\,\chi_p - 1/6\, R^d_p\,\chi_q\right] \\
& + \bar B(\chi_p,\chi_p;0)\, L^r_6 \left[4\,\bar\chi_1\bar\chi^{pp}_{\eta\eta 1} + 8\,\bar\chi_1 R^p_{q\eta}\,\chi_p - 8\,\bar\chi_1 R^c_p\,\chi_p\right] + 4\,\bar B(\chi_p,\chi_p;0)\, L^r_7\,(R^d_p)^2 \\
& + \bar B(\chi_p,\chi_p;0)\, L^r_8 \left[4/3\,\bar\chi^{pp}_{\eta\eta 2} + 8/3\, R^p_{q\eta}\,\chi_p^2 - 8/3\, R^c_p\,\chi_p^2\right] + \bar B(\chi_p,\chi_p;0)^2 \left[-1/18\, R^p_{q\eta} R^d_p\,\chi_p + 1/18\, R^c_p R^d_p\,\chi_p\right. \\
& \left. + 1/288\,(R^d_p)^2\right] + 1/18\,\bar B(\chi_p,\chi_p;0)\bar B(\chi_p,\chi_\eta;0) \left[R^\eta_{pp} R^d_p\,\chi_p - R^\eta_{13} R^d_p\,\chi_p\right]
\end{aligned}
$$

plus several more pages

Why so long expressions

- Many different quark and meson masses $(\chi_{ij} = B_0(m_i + m_j))$
- Charged propagators: $-i\, G^c_{ij}(k) = \frac{\epsilon_j}{k^2 - \chi_{ij} + i\varepsilon} \quad (i \neq j)$
- Neutral propagators: $G^n_{ij}(k) = G^c_{ij}(k)\, \delta_{ij} - \frac{1}{n_{\rm sea}}\, G^q_{ij}(k)$

$$-i\, G^q_{ii}(k) = \frac{R^d_i}{(k^2 - \chi_i + i\varepsilon)^2} + \frac{R^c_i}{k^2 - \chi_i + i\varepsilon} + \frac{R^\pi_{\eta ii}}{k^2 - \chi_\pi + i\varepsilon} + \frac{R^\eta_{\pi ii}}{k^2 - \chi_\eta + i\varepsilon}$$

$$R^i_{jkl} = R^z_{i456jkl}, \qquad R^d_i = R^z_{i456\pi\eta},$$

$$R^c_i = R^i_{4\pi\eta} + R^i_{5\pi\eta} + R^i_{6\pi\eta} - R^i_{\pi\eta\eta} - R^i_{\pi\pi\eta}$$

$$R^z_{ab} = \chi_a - \chi_b, \qquad R^z_{abc} = \frac{\chi_a - \chi_b}{\chi_a - \chi_c}, \qquad R^z_{abcd} = \frac{(\chi_a - \chi_b)(\chi_a - \chi_c)}{\chi_a - \chi_d}$$

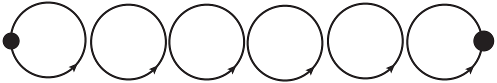
$$R^z_{abcdefg} = \frac{(\chi_a - \chi_b)(\chi_a - \chi_c)(\chi_a - \chi_d)}{(\chi_a - \chi_e)(\chi_a - \chi_f)(\chi_a - \chi_g)}$$

- Relations $\Longrightarrow$ order of magnitude smaller

Double poles ?

Think quark lines and add gluons everywhere

Full



Quenched

So no resummation at the quark level:

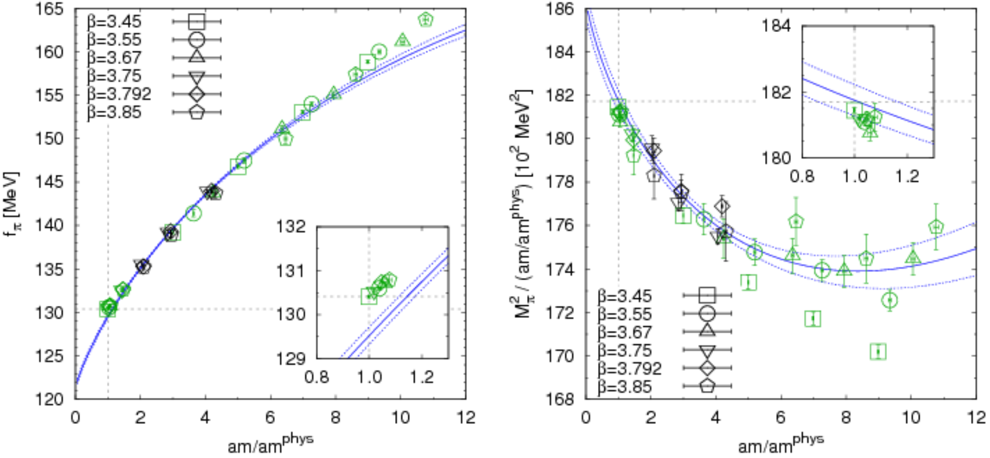
naively a double pole

Same follows from inverting the lowest order kinetic terms

Leads to enhanced chiral logarithms without M^2 in front

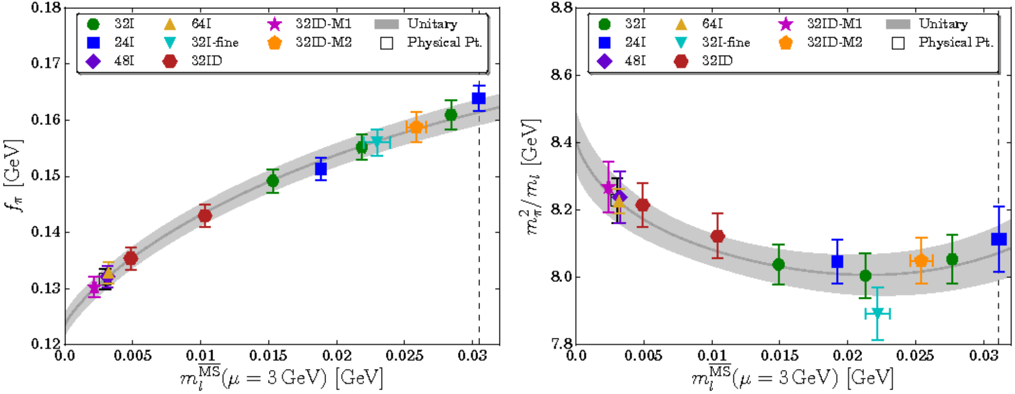
Some examples

S. Borsanyi et al,Phys. Rev. D **88** (2013), 014513 [arXiv:1205.0788 [hep-lat]].



Some examples

P. A. Boyle, *et al.* Phys. Rev. D **93** (2016) no.5, 054502 [arXiv:1511.01950 [hep-lat]].



Conclusions

- Short intro to ChPT
- Named a lot extensions and possibilities
- Mass and decay constant at 3-loops and the lattice
- Finite volume
- Partially quenched