CHIRAL LAGRANGIANS AT HIGHER ORDERS

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Why is this a difficult problem

- Why is this so difficult?
  - "Just write down all possible terms"
  - Same theory can look very different
  - Redundant terms
- Simple example
  - Take an $O(N)$ symmetric free field theory: $\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$
  - $g \in O(N)$: $\Phi \to g\Phi$
  - $\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi - \frac{1}{2}\Phi^T \Phi$
  - Describes N noninteracting scalars of the same mass m

Why is this a difficult problem: example

- Take an $O(N)$ symmetric field theory: $\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$
- $g \in O(N)$: $\Psi \to g\Psi$
- Take the Lagrangian

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\partial_\mu \Psi^T \partial^\mu \Psi - \frac{1}{2} m^2 \Psi^T \Psi + \lambda \partial_\mu \Psi^T \partial^\mu \Psi \Psi^T \Psi - \lambda m^2 \left(\Psi^T \Psi\right)^2 \\ &+ \frac{1}{2}\lambda^2 \partial_\mu \Psi^T \partial^\mu \Psi \left(\Psi^T \Psi\right)^2 + 4\lambda^2 \Psi^T \partial_\mu \Psi \Psi^T \partial^\mu \Psi \Psi^T \Psi - \frac{1}{2}\lambda^2 m^2 \left(\Psi^T \Psi\right)^3\end{aligned}$$

- Describes N noninteracting scalars of the same mass m
- Check it with Feynman diagrams if you like

Why is this a difficult problem: example

- Why are these two the same?
  - The two are related by $\Phi = \Psi\left(1 + \lambda\Psi^T\Psi\right)$
  - Theorem in field theory: field redefinitions do not change the physics
- Why didn't you hear about it in (introductory) field theory?
  - In (obviously) renormalizable field theory: very little allowed
  - But also here: explains why CKM matrix with 9 parameters only gives you three mixing angles and one phase
- And there are even more things you can do
- Important since in effective field theories you want to know how many free parameters you have at a given order

Exploring the consequences of
the chiral symmetry of QCD
and its spontaneous breaking
using effective field theory techniques

Derivation from QCD:
H. Leutwyler,
On The Foundations Of Chiral Perturbation Theory,
Ann. Phys. 235 (1994) 165 [hep-ph/9311274]

For references to lectures see: `http://www.thep.lu.se/~bijnens/chpt.html`

Chiral Perturbation Theory

A general Effective Field Theory:

- Relevant degrees of freedom
- A powercounting principle (predictivity)
- Has a certain range of validity

Chiral Perturbation Theory:

- Degrees of freedom: Goldstone Bosons from spontaneous breaking of chiral symmetry
- Powercounting: Dimensional counting in momenta/masses
- Breakdown scale: Resonances, so about M_ρ.

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Chiral Symmetry

QCD: N_f light quarks: equal mass: interchange: $g_V \in SU(N_f)_V$:

$$q = \begin{pmatrix} q_1 \\ \vdots \\ q_{N_f} \end{pmatrix} \to g_V q$$

But $$\mathcal{L}_{QCD} = \sum_{q=u,d,\ldots} \left[i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q \left(\bar{q}_R q_L + \bar{q}_L q_R \right) \right]$$

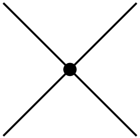
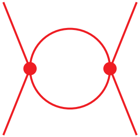
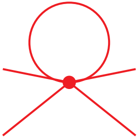
So if $m_q = 0$ then $SU(N_f)_L \times SU(N_f)_R$.

Spontaneous breakdown

- $\langle \bar{q} q \rangle = \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle \neq 0$
- $SU(N_f)_L \times SU(N_f)_R$ broken spontaneously to $SU(N_f)_V$
- $N_f(N_f - 1)$ generators broken $\Longrightarrow$ $N_f(N_f - 1)$ massless degrees of freedom and interaction vanishes at zero momentum

Goldstone Bosons

Power counting in momenta: Meson loops, Weinberg powercounting

rules		one loop example	
	p^2		$(p^2)^2\,(1/p^2)^2\,p^4 = p^4$
	$1/p^2$		$(p^2)\,(1/p^2)\,p^4 = p^4$
$\int d^4p$	p^4		

Goldstone Bosons

- Full symmetry group: G
- Unbroken symmetry group: H
- Goldstone bosons live on the coset G/H
- I will talk about two cases only:
  - $SU(N_f) \times SU(N_f)/SU(N_f)$ with $G/H \approx SU(N_f)$
  - $SO(N)/SO(N-1)$ with $G/H = S^{N-1}$ (surface of N-dimensional sphere)
- Parametrize by
  - $U = \exp\left(\frac{i\sqrt{2}}{F} M\right)$ with $U^\dagger U = 1$
    $U \to g_R U g_L^\dagger$ for $(g_L, g_R) \in SU(N)_L \times SU(N)_R$
    $M = \phi^i T^i$ and T^i the generators of $SU(N)$
    ϕ^i are the $N(N-1)$ are the Goldstone Boson fields
  - $\Phi^T = (\phi_0 \ \ldots \ \phi_{N-1})/F$ with $\Phi^T \Phi = 1$
    $\Phi \to g\Phi$ for $g \in SO(N)$
    $\phi_1, \ldots, \phi_{N-1}$ are the Goldstone Boson fields

External field and spurions: example

- Quark masses in the Lagrangian: $q^T = (u\ d\ s)$
  $-\sum_{q=u,d,s} m_q \left(\overline{q}_L q_R + \overline{q}_R q_L\right) =$
  $-q_L \mathrm{diag}(m_u, m_d, m_s) q_R - q_R \mathrm{diag}(m_u, m_d, m_s) q_L$
- Not invariant under chiral symmetry
- Make it invariant by defining a *spurion* (field) $X \to g_R X g_L^\dagger$
- Write in QCD the term $-\overline{q}_R X q_L - \overline{q}_L X^\dagger q_R$ instead of quark masses
- Use now in the low-energy effective theory both U and $\chi = 2B_0 X$
- Method can be generalized to other cases

External field method

- Problem: Ward identities for fields that transform nonlinearly
- Solution: Gasser, Leutwyler 84,85: use external field method and generate Green functions of QCD currents/densities from those
- with $q^T = (u\ d\ s \cdots)$:

$$\mathcal{L}_{QCD} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \overline{q} i\gamma^\mu \left(D_\mu - i v_\mu - i a_\mu \gamma_5\right) q - \overline{q} s q + \overline{q} i \gamma_5 p q$$

- v_μ, a_μ, s, p are $N_f \times N_f$ matrices: the external fields
- Chiral symmetry made local $g_L, g_R \in SU(N_f)_L \times SU(N_f)_R$

$$q_{L,R} \longrightarrow g_{L,R} q_{L,R}, \qquad X = s + ip \longrightarrow g_R (s + ip) g_L^\dagger$$

$$\ell_\mu \equiv v_\mu - a_\mu \longrightarrow g_L \ell_\mu g_L^\dagger - i\partial_\mu g_L g_L^\dagger, \quad r_\mu \equiv v_\mu + a_\mu \longrightarrow g_R r_\mu g_R^\dagger - i\partial_\mu g_R g_R^\dagger$$

- s, p, ℓ_μ, r_μ: external fields

External field method

- Define Green functions of QCD currents by functional derivatives w.r.t. the external fields of
  $$Z_{QCD}(v_\mu, a_\mu, s, p) = \int [dq d\overline{q} dG] \exp\left(i \int d^4x \mathcal{L}_{QCD}\right)$$
- Put in photons in v_μ, quark masses in s,... by comparing with the Lagrangian withose parts included
- If dealing with other operators: add more external fields (spurions)
- Now write theory with the Goldstone bosons ϕ^a:
  $Z_{ChPT}(v_\mu, a_\mu, s, p) = \int [d\phi^a] \exp\left(i \int d^4x \mathcal{L}_{ChPT}\right)$
- $\mathcal{L}_{ChPT}$ has the same (chiral) symmetries as $\mathcal{L}_{QCD}$
- Finally (proof follows from all singularities at low energies included this way, the remainder can be Taylor expanded)
  $Z_{QCD}(v_\mu, a_\mu, s, p) \approx Z_{ChPT}(v_\mu, a_\mu, s, p)$

Building blocks: $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$

- We can now use a somewhat different notation: advantage only transformations under the unbroken subgroup needed
- Callan, Coleman, Wess, Zumino 1969
- $u = \exp i\phi/(\sqrt{2}F) = \sqrt{U} \longrightarrow \; g_R u h^\dagger = h u g_L^\dagger$
- $u_\mu \equiv i\left(u^\dagger(\partial - i r_\mu)u - u(\partial_\mu - i\ell_\mu)u^\dagger\right) \longrightarrow \; h u_\mu h^\dagger$
- $\Gamma_\mu = \frac{1}{2}\left(u^\dagger(\partial - i r_\mu)u + u(\partial_\mu - i\ell_\mu)u\dagger\right) \longrightarrow \; h\Gamma_\mu h^\dagger - \partial_\mu h h^\dagger$
- Γ_μ can be used to define a covariant derivative

Building blocks: $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$

- $\nabla_\mu \Psi = \partial_\mu \Psi + \Gamma_\mu \Psi$ for $\Psi \to h\Psi$
- $\nabla_\mu X = \partial_\mu X + [\Gamma_\mu, X]$ for $X \to hXh^\dagger$
- $\chi \equiv 2B_0(s+ip) \longrightarrow g_R \chi g_L^\dagger$
- $F_{L\mu\nu} = \partial_\mu \ell_\nu - \partial_\nu \ell_\mu - i\,[\ell_\mu, \ell_\nu] \longrightarrow g_L F_{L\mu\nu} g_L^\dagger$
- $F_{R\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - i\,[r_\mu, r_\nu] \longrightarrow g_R F_{R\mu\nu} g_R^\dagger$
- $\chi_\pm \equiv u^\dagger \chi u^\dagger \pm u \chi^\dagger u$
- $f_{\pm\mu\nu} = u F_{L\mu\nu} u^\dagger \pm u^\dagger F_{R\mu\nu} u$
- Final building blocks all go as $X \longrightarrow hXh^\dagger$:
  Order p^1: u_μ, ∇_μ; order p^2 $\chi_\pm, f_{\pm\mu\nu}$
- $\langle u_\mu \rangle = \langle f_{\pm\mu\nu} \rangle = 0$
- Other choices, purely left-handed,... transformations are possible

Building blocks: $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$

- Transformations under discrete symmetries

	P	C	h.c.
u_μ	$-\varepsilon(\mu)u_\mu$	u_μ^T	u_μ
$\chi_\pm$	$\pm\chi_\pm$	$\chi_\pm^T$	$\pm\chi_\pm$
$f_{\pm\mu\nu}$	$\pm\varepsilon(\mu)\varepsilon(\nu)f_{\pm\mu\nu}$	$\mp f_{\pm\mu\nu}^T$	$f_{\pm\mu\nu}$

$\varepsilon(0) = -\varepsilon(i = 1, 2, 3) = 1.$

Lagrangians: Lowest order

- $N_f = 2$ and add for $N_f = 3$ $\phi(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\,\eta_8}{\sqrt{6}} \end{pmatrix}$.

- p^0: no building block exists
- LO or p^2: $\langle u_\mu u^\mu \rangle$, $\langle \nabla^\mu u_\mu \rangle$, $\langle \chi_+ \rangle$, $\langle \chi_- \rangle$,
  use P and $\langle u_\mu \rangle = 0$
  $\mathcal{L}_2 = \frac{F_0^2}{4} \left[\langle u_\mu u^\mu \rangle + \langle \chi_+ \rangle \right]$
- Usually in terms of $U = u^2 \longrightarrow g_R U g_L^\dagger$ and $D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu$,
- $\mathcal{L}_2 = \frac{F_0^2}{4} \left[\left\langle D_\mu U D^\mu U^\dagger \right\rangle + \left\langle \chi U^\dagger + U \chi^\dagger \right\rangle \right]$

Lagrangians: Lagrangian structure

	2 flavour		3 flavour		PQChPT/N_f flavour	
p^2	F, B	2	F_0, B_0	2	F_0, B_0	2
p^4	l_i^r, h_i^r	7+3	L_i^r, H_i^r	10+2	$\hat{L}_i^r, \hat{H}_i^r$	11+2
p^6	c_i^r	52+4	C_i^r	90+4	K_i^r	112+3

p^2: Weinberg 1966

p^4: Gasser, Leutwyler 84,85

p^6: JB, Colangelo, Ecker 99,00

- L_i LEC = Low Energy Constants = ChPT parameters
- H_i: contact terms: value depends on definition of currents/densities
- Finite volume: no new LECs
- Other effects: (many) new LECs
- Many extensions classified: $\varepsilon_{\mu\nu\alpha\beta}$, weak decays,...

P, C Hermitian conjugate (H)

- X_i: a building block
- P: enforce by having an even number of parity-odd blocks (we assume no $\epsilon_{\mu\nu\alpha\beta}$)
- C and H relate the same building blocks

$$
\begin{aligned}
C\left(\langle X_1 \dots X_n\rangle\right) &= \pm\langle X_1^T \dots X_n^T\rangle = \pm\langle X_n \dots X_1\rangle\,,\\
\left(\langle X_1 \dots X_n\rangle\right)^\dagger &= \langle X_n^\dagger \dots X_1^\dagger\rangle = \pm\langle X_n \dots X_1\rangle\,,
\end{aligned}
$$

- $\mathcal{O}_i \longrightarrow \lambda_\pm^C\,\lambda_\pm^{\mathrm{h.c.}}\,\mathcal{O}_j$ look at $(\pm,\pm)$
- $i = j$

$$
\begin{aligned}
(+,+):&\quad \mathcal{O}_i = \mathcal{O}_i^+ & (-,+):&\quad \mathcal{O}_i = \mathcal{O}_i^-\,,\\
(+,-):&\quad \mathcal{O}_i = i\mathcal{O}_i^+ & (-,-):&\quad \mathcal{O}_i = i\mathcal{O}_i^-\,.
\end{aligned}
$$

P, C Hermitian conjugate (H)

- $j \neq i$

$$
\begin{aligned}
(+,+):\quad & \mathcal{O}_i = \frac{\mathcal{O}_i^+ + i\mathcal{O}_i^-}{2}\,, \quad \mathcal{O}_j = \frac{\mathcal{O}_i^+ - i\mathcal{O}_i^-}{2}\,,\\
(-,+):\quad & \mathcal{O}_i = \frac{i\mathcal{O}_i^+ + \mathcal{O}_i^-}{2}\,, \quad \mathcal{O}_j = \frac{-i\mathcal{O}_i^+ + \mathcal{O}_i^-}{2}\,,\\
(+,-):\quad & \mathcal{O}_i = \frac{i\mathcal{O}_i^+ + \mathcal{O}_i^-}{2}\,, \quad \mathcal{O}_j = \frac{i\mathcal{O}_i^+ - \mathcal{O}_i^-}{2}\,,\\
(-,-):\quad & \mathcal{O}_i = \frac{\mathcal{O}_i^+ + i\mathcal{O}_i^-}{2}\,, \quad \mathcal{O}_j = \frac{-\mathcal{O}_i^+ + i\mathcal{O}_i^-}{2}\,.
\end{aligned}
$$

- The final Lagrangian should only contain the monomials $\mathcal{O}_i^+$.

Constraints

- p^2, p^4, p^6 were done essentially by hand
- p^8 way too many terms for that
- use FORM
- Cyclicity: use cyclic functions
- Check all ways of relabelling indices explicitly
  use Python to rewrite FORM output back into FORM commands
- Easier to work with single term operators for all relations: rewriting in $\mathcal{O}_i^+$ done at the end
- Done also for the anomalous case (like needed for $\pi^0 \to \gamma\gamma$) up to p^8

Contact terms

- Building blocks: $\chi, \chi^\dagger, F_{L\mu\nu}, F_{R\mu\nu}$
- all under same simple group lost
- Covariant derivatives:

$$D_\mu\chi = \partial_\mu\chi - ir_\mu\chi + i\chi\ell_\mu$$
$$D_\rho F_{L\mu\nu} = \partial_\rho F_{L\mu\nu} - i\ell_\rho\chi + iF_{L\mu\nu}\ell_\rho$$
$$D_\rho F_{R\mu\nu} = \partial_\rho F_{R\mu\nu} - ir_\rho\chi + iF_{R\mu\nu}r_\rho$$

- P, C, H more tricky as well
- If finite N_f: (Kaplan-Manohar)
  new operator of order p^{2N_f-2}, not singular for $\chi \to 0$
  $\tilde{\chi} \equiv \left(\det(\chi)\chi^{-1}\right)^\dagger \longrightarrow g_R\tilde{\chi}g_L^\dagger\,.$

$$N_f = 2: \qquad \tilde{\chi} = \begin{pmatrix} x_{22}^* & -x_{21}^* \\ -x_{12}^* & x_{11}^* \end{pmatrix},$$

$$N_f = 3: \qquad \tilde{\chi} = \begin{pmatrix} x_{22}^*x_{33}^* - x_{23}^*x_{32}^* & x_{31}^*x_{23}^* - x_{21}^*x_{33}^* & x_{21}^*x_{32}^* - x_{31}^*x_{22}^* \\ x_{32}^*x_{13}^* - x_{12}^*x_{33}^* & x_{11}^*x_{33}^* - x_{31}^*x_{13}^* & x_{31}^*x_{12}^* - x_{11}^*x_{32}^* \\ x_{12}^*x_{23}^* - x_{22}^*x_{13}^* & x_{21}^*x_{13}^* - x_{11}^*x_{23}^* & x_{11}^*x_{22}^* - x_{12}^*x_{21}^* \end{pmatrix}$$

Sources of relations

- Partial integration/total derivatives
- Terms that can be removed by LO EOM/field redefinitions
- "Commuting of partial derivatives"
- Bianchi identity
- Cayley Hamilton (for finite N_f)
- Schouten identity

Partial integration/Total derivatives

- Partial integration can lead to very different looking terms
- Main problem: how to make sure we have all of them
- Solution: each partial derivative relation corresponds to a total derivative
- Classify all invariant monomials as before but now with one free Lorentz index
- Take ∂^μ of those and it gives all partial integration relations
- Example:

$$
\begin{aligned}
0 &= \partial^\mu \langle \nabla_\mu u_\nu u^\nu u_\alpha u_\alpha \rangle \\
&= \langle \nabla^\mu \nabla_\mu u_\nu u^\nu u_\alpha u_\alpha \rangle + \langle \nabla_\mu u_\nu \nabla^\mu u^\nu u_\alpha u_\alpha \rangle \\
&\quad + \langle \nabla_\mu u_\nu u^\nu \nabla^\mu u_\alpha u_\alpha \rangle + \langle \nabla_\mu u_\nu u^\nu u_\alpha \nabla^\mu u_\alpha \rangle
\end{aligned}
$$

- Having all relations allows for many simplifications later

Field redefinitions – LO equations of motion

- The S-matrix does not change under a field redefinition: $\phi = \phi' + F(\phi')$ with $F(x \to 0) \to 0$ fast enough.
- In the functional integral: "just" a change of variables
- For classifying a Lagrangian: equivalent to removing "equation of motion terms"
- Simple explanation in the one-flavour case
- Works also if symmetries present
- Need a concept of power-counting or otherwise ordering

Field redefinitions – LO equations of motion

Use g to indicate orders

$$\mathcal{L} = \left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V_0(\phi)\right) + g\left[\left(\partial^2\phi + V_0'(\phi)\right)V_{1EOM}(\phi,\partial\phi) + V_1(\phi,\partial\phi)\right] + \mathcal{O}(g^2)$$

Now define $\phi = \phi' + gV_{1EOM}(\phi',\partial\phi')$

$$\mathcal{L} = \left(\frac{1}{2}\partial^\mu\phi'\partial_\mu\phi' - V_0(\phi')\right) + g\left(\partial^\mu\phi\partial_\mu(V_0(\phi') - V_0'(\phi')\right)V_{1EOM}(\phi',\partial\phi') + g\left(\partial^2\phi' + V_0'(\phi')\right)V_{1EOM}(\phi',\partial\phi') + gV_1(\phi',\partial\phi') + \mathcal{O}(g^2)$$

Note: $\mathcal{O}(g^2)$ changed

After partial integration: EOM terms at $\mathcal{O}(g)$ cancel but changes at higher orders:

using EOM OK for classifying terms, not for doing calculations

Field redefinitions vs EOM

- Do order by order (include g^2 changes from step 1)
  $\phi' = \phi'' + g^2 V_{2EOM}(\phi'', \partial\phi'')$
- More than one field also works: use
  $\phi^a = \phi^{a\prime} + g V_1^a(\{\phi^{b\prime}, \partial\phi^{b\prime}\})$
- What about symmetries?
  - $\mathcal{L} = \mathcal{L}_0 + g\mathcal{L}_1 + g^2\mathcal{L}_2 + \cdots$
  - Each $\mathcal{L}_i$ is invariant under the symmetry
  - EOM^a is derived by $\phi^a \to \phi^a + \delta\phi^a$ where $\delta\phi^a$ must be compatible with the symmetry
  - $\mathcal{L}_0 \to \mathcal{L}_0 + (\delta\phi^a \mathrm{EOM}^a)$ means that $\delta\phi^a \mathrm{EOM}^a$ is invariant under the symmetry
  - EOM terms in $\mathcal{L}_i$ are invariant under the symmetry and of the form EOM$^a V_i^a(\{\phi^b, \partial\phi^b\})$
  - $\Longrightarrow$ $\phi^a = \phi' + g^i V_i^a(\{\phi^{b\prime}, \partial\phi^{b\prime}\})$ are field transformations compatible with the symmetry that remove the EOM terms

Field redefinitions – LO equations of motion

- So we use
$$\nabla^\mu u_\mu - \frac{i}{2}\left(\chi_- - \frac{1}{N_f}\langle\chi_-\rangle\right) = 0\,.$$
- take all operators, look for $\nabla^\mu u_\mu$ and replace by above to get a relation
- Example
operator: $\langle \chi_+ u^\rho \chi_+ \nabla_\rho \nabla^\mu u_\mu\rangle$
relation: $0 = \langle \chi_+ u^\rho \chi_+ \nabla_\rho \nabla^\mu u_\mu\rangle - \frac{i}{2}\langle \chi_+ u^\rho \chi_+ \nabla_\rho \chi_-\rangle$
$+\frac{i}{2N_f}\langle \chi_+ u^\rho \chi_+\rangle\langle \nabla_\rho \chi_-\rangle$
- Since we have all operators, all partial integrations and all "commuting": only need to do it for "naked" $\nabla^\mu u_\mu$.

"Commuting of partial derivatives"/Bianchi

- Commuting:
  - $f_{-\mu\nu} - \nabla_\nu u_\mu + \nabla_\mu u_\nu = 0$
  - $[\nabla_\mu, \nabla_\nu] X = [\Gamma_{\mu\nu}, X]$
  - $\Gamma_{\mu\nu} = \frac{1}{4}[u_\mu, u_\nu] - \frac{i}{2} f_{+\mu\nu}$
- Bianchi
  - $B_{\mu\nu\rho} \equiv \nabla_\mu \Gamma_{\nu\rho} + \nabla_\nu \Gamma_{\rho\mu} + \nabla_\rho \Gamma_{\mu\nu} = 0\,.$
    $$B_{\mu\nu\rho} = \frac{1}{4}\Big([u_\rho, f_{-\mu\nu}] + [u_\mu, f_{-\nu\rho}] + [u_\nu, f_{-\rho\mu}]\Big) - \frac{i}{2}\Big(\nabla_\rho f_{+\mu\nu} + \nabla_\mu f_{+\nu\rho} + \nabla_\nu f_{+\rho\mu}\Big)$$
  - Generate all terms including $B_{\mu\nu\lambda}$
  - $\nabla_\rho B_{\mu\nu\lambda}$ not needed: we have all p.i. relations
  - $D_\mu F_{L\nu\rho} + D_\mu F_{L\nu\rho} + D_\mu F_{L\nu\rho} = 0$ (and $L \leftrightarrow R$) follow from "Commuting" and Bianchi for $\Gamma_{\mu\nu}$

Cayley-Hamilton relations

- Any matrix A satistfies its own characteristic polynomial:
  $p(\lambda) \equiv \det(\lambda I - A)\,, \qquad \text{and} \qquad p(A) = 0\,.$
- Expand in $1/\lambda$ using $p(\lambda) = \lambda^n \exp\{\mathrm{tr}\,[\ln(I - A/\lambda)]\}$
  The expansion has no terms negative in λ and stops at λ^n
- This leads to the relations

$$
\begin{aligned}
n = 1 &: A - I\langle A\rangle = 0\\
n = 2 &: A^2 - A\langle A\rangle - \frac{1}{2}I\langle A^2\rangle + \frac{1}{2}I\langle A\rangle^2 = 0\\
n = 3 &: A^3 - A^2\langle A\rangle - \frac{1}{2}A\langle A^2\rangle + \frac{1}{2}A\langle A\rangle^2 - \frac{1}{3}I\langle A^3\rangle\\
&\quad + \frac{1}{2}I\langle A\rangle\langle A^2\rangle - \frac{1}{6}I\langle A\rangle^3 = 0
\end{aligned}
$$

- The last terms with I always are related to $\det(A)$.

Cayley-Hamilton relations

- Make more useful by $A = B + C + \cdots$
- $n = 2$: $A = B + C$, only keep terms with B and C:
  $\{B, C\} = B\langle C\rangle + C\langle B\rangle + \langle BC\rangle - \langle B\rangle\langle C\rangle$.
- $n = 3$ and $\langle B\rangle = \langle C\rangle = \langle D\rangle = \langle E\rangle = 0$ use $A = B + C + D$, multiply by E and take trace

$$\sum_{6\ perm} \langle BCDE\rangle = \sum_{3\ perm} \langle BC\rangle\langle DE\rangle \ .$$

- Simply taking the trace of the relations does not give any new results, that is satisfied automatically.
- When implementing: use $B, C, \ldots$ as building blocks and products of building blocks

Schouten identity

- No fully antisymmetric tensor with five indices in four dimensions
- We have assumed no $\epsilon_{\mu\nu\alpha\beta}$: only relevant when have 5 different indices, so just from p^{10}
- Very relevant for the anomalous case including $\epsilon_{\mu\nu\alpha\beta}$
- It's never clear whether more relations exist in this way
  - can you determine all LECs from explicit Green functions (or experiment)
  - Number of terms can be determined from "Hilbert series" (very mathematical)

What to keep

Preferentially keep/remove (as much as possible):

1. Keep maximal number of independent contact terms
2. Remove terms that vanish when external fields vanish
3. Remove terms with covariant derivatives in favour of those involving external fields.
4. Remove terms that contribute to processes with a low number of mesons, count occurrences of u_μ, χ_- and $f_{-\mu\nu}$.
5. Scalar-pseudoscalar external fields are placed before those with only vector-axial-vector external fields.
6. Keep terms with lower number of flavour traces. This is to make the large N_c counting of the monomials explicit, only leading in N_c is equivalent to keeping only single trace monomials.

Still leaves a choice to be made which to keep

Technically

- FORM for the main part: generating all terms and relations.
- equivalent terms produced by FORM, using Python rewritten back into FORM.
- Identical relations removed using FORM
- Final number of independent relations done with Gaussian elimination (sparse matrix methods) with gmp exact arithmetic
- Main restriction: memory size, not CPU time (but grows very fast with order)
- Up to 50 000 monomials, 200 000 relations
- JB, Nils Hermansson-Truedsson, Si Wang, JHEP 01 (2019) 102 [1810.06834] p^8 normal
- JB, Nils Hermansson-Truedsson Joan Ruiz-Vidal, JHEP 01(2024)009 [12310.2054] p^8 anomalous
- JB, Sven Bjarke Gudnason, Jiahui Yu, Tiantian Zhang, JHEP 05 (2023) 061 [2212.07901]
  Testing Hilbert series for $O(N)$ model in many dimensions and up to p^{12}

Results: p^8 normal all

	N_f		$N_f = 3$		$N_f = 2$	
	Total	Contact	Total	Contact	Total	Contact
p^2	2	0	2	0	2	0
p^4	13	2	12	2	10	3
p^6	115	3	94	4	56	4
p^8	1862	22	1254	21	475	23

Table: Number of monomials in the minimal basis for the case with all external fields included. Also listed is how many of them are contact terms. Our results agree with the known ones for p^2, p^4, p^6.

Results: p^8 normal only vector-axial-vector

	N_f		$N_f = 3$		$N_f = 2$	
	Total	Contact	Total	Contact	Total	Contact
p^2	1	0	1	0	1	0
p^4	7	1	6	1	5	1
p^6	59	2	44	2	27	2
p^8	963	15	591	13	238	11

Table: Number of monomials in the minimal basis for the case with no scalar or pseudoscalar external fields included. Also listed is how many of them are contact terms.

Note: tested using Green functions at p^6

P. Ruiz-Femenía and M. Zahiri-Abyaneh, 1507.00269

Results: p^8 normal no external fields; by meson number

	#mesons	N_f	$N_f = 3$	$N_f = 2$
p^2	4	1	1	1
p^4	4	4	3	2
p^6	4	4	3	2
	6	15	8	3
p^8	4	6	5	3
	6	60	31	9
	8	69	20	4

Table: Number of monomials in the minimal basis for the case with no external fields included that produce vertices starting at the given number of mesons.

Results: p^8 anomalous

$\langle f_+^{\mu\nu} \rangle = 0$	N_f	$N_f = 5$	$N_f = 4$	$N_f = 3$	$N_f = 2$
	Total	Total	Total	Total	Total
Full	999	998	950	705	92
No $\chi_\pm$	565	564	525	369	0
No $f_\pm^{\mu\nu}$	79	79	73	45	2
Only u_μ	36	36	31	16	0

Number of monomials in the obtained minimal basis. Case $\langle f_+^{\mu\nu} \rangle = 0$, which means that the singlet non-zero trace physically relevant for $N_f = 2$ is not included.

Results: p^8 anomalous

$\langle f_+^{\mu\nu} \rangle \neq 0$	N_f	$N_f = 5$	$N_f = 4$	$N_f = 3$	$N_f = 2$
Full	1210	1209	1161	892	211
No $\chi_\pm$	702	701	662	486	77
No $f_\pm^{\mu\nu}$	79	79	73	45	2
Only u_μ	36	36	31	16	0

Number of monomials in the obtained minimal basis when $\langle f_+^{\mu\nu} \rangle \neq 0$. Again we include the cases where only a subset of the external field building blocks contribute.

Hilbert series

- Uses group theory and conformal field theory to obtain the number of operators at a given level
- "plethystic exponential"
- By summing characters of a representation over all group elements you can get the dimension of the representation
- L. Graf *et al., 2, 12, 117, 1959, 45171, 1170086, ..., JHEP 01 (2021) 142 [2009.01239]*
- Agreed with our p^8
- Can this be checked more?
- Yes: do the $O(N)$ model in various dimensions and up to p^{12} by also explicitly constructing the Lagrangians
- JB, Sven Bjarke Gudnason, Jiahui Yu, Tiantian Zhang, JHEP 05 (2023) 061 [2212.07901]
  Testing Hilbert series for $O(N)$ model in many dimensions and up to p^{12}

$O(N)$ model Hilbert series and explicit construction

n_d	D	N	#terms
2	≥2	≥2	1
4	≥2	2	1
		≥3	2
6	2	2	1
		≥3	3
	≥3	2	2
		3	4
		≥4	5
8	2	2	3
		3	8
		≥4	9
	3	2	4
		3	12
		≥4	15
	≥4	2	4
		3	13
		4	16
		≥5	17

n_d	D	N	#terms
10	2	2	3
		3	14
		≥4	16
	3	2	7
		3	34
		4	48
		≥5	49
	4	2	7
		3	38
		4	55
		≥5	58
	≥5	2	8
		3	39
		4	57
		5	60
		≥6	61

n_d	D	N	#terms
12	2	2	7
		3	34
		4	45
		≥5	46
	3	2	17
		3	114
		4	185
		≥5	193
	4	2	20
		3	147
		4	253
		5	275
		≥6	276
	5	2	21
		3	153
		4	264
		5	289
		≥6	292
	≥6	2	21
		3	154
		4	265
		5	291
		6	294
		≥7	295

Table: Type 1 # terms up to $n_d = 12$, Hilbert series method and explicit construction method

Conclusions

- Constructing higher order Lagrangians is possible but not entirely trivial, especially to get a minimal one.
- Hilbert series allows for determining the number of terms independently
- Ongoing: constructing higher order Lagrangians including nonleptonic weak and electromagnetic interactions (more spurions)