Exercise 1 - Characteristic dark matter (DM) speeds

Estimate the characteristic speed of dark matter particles bound to the dark halos of several structures in the Universe using the Virial Theorem

$$m v^2 \simeq \frac{G M_{halo} m}{R_{halo}}$$

m: DM particle mass ; v: characteristic speed ; M_{halo}, R_{halo} = mass and radius of the structure (including DM)

Estimate the speed v for 1) a dwarf galaxy (typical masses and radius of DG are $M_{DG} \simeq 2\times10^6 M_\odot$, $R_{DG} \simeq 1$ kp) 2) the Milky Way (or other galaxies, $M_{MW} \simeq 10^{12} M_\odot$, $R_{MW} \simeq 100$ kp) and 3) galaxy cluster (with, say, $M_C \simeq 3\times10^{14} M_\odot$, $R_C \simeq 2$ Mpc). Those just quoted are approximate values for the dark halos of the mentioned structures.

Help: you will find very useful to write first Newton's constant G in units of $\frac{\text{Mpc}\ \text{km}^2}{M_\odot\ \text{s}^2}$. Here

$M_\odot$ is a solar mass, $M_\odot = 2.99\times10^{30}$ kg $\simeq 1.7\times10^{57}$ GeV, and pc is a parsec, 1 pc = $3.086\ 10^{16}$ m = 3.26 ly.

Notice that in all equations in my lectures I use "Natural Units" $c=1$, $\hbar=1$ in which

$$1\,\text{GeV} = 1.8\times10^{-27}\,\text{kg} = \frac{1}{0.197\ 10^{-15}\,\text{m}} = \frac{1}{6.58\ 10^{-25}\,\text{s}}$$

Exercise 2 - Self Interacting Dark Matter (SIDM)

SIDM must have a cross section very close to its upper limit in a particular type of structure to be substantially different from the usual collision-less CDM. The ratio $\left(\frac{\sigma_{max}}{m} \Big/ 6 \times 10^{-25} \frac{cm^2}{GeV} \right)$ is between a few - to 100 to have an effective core creation in dwarf galaxies, but is $\simeq 1$ in clusters (here σ_{max} is the maximum value of the self-interaction cross section). A constant cross section $\sigma_{self} = \sigma_{max}$ which would be effective at dwarf galaxy scales would then be forbidden by the limits coming from galaxy clusters. But this may not be a problem if the cross section depends on the DM speed.

Using the results you obtained in Excercise 1 say

2.1 Which dependence of the self scattering cross-section on the DM particles speed could give σ_{self} in dwarf galaxies $\simeq 100 \times \sigma_{self}$ in galaxy clusters?

2.2 Probably you know that in Rutherford scattering the cross section is inversely proportional to the square of the kinetic energy, i.e. $\sigma \sim 1/v^4$. This is characteristic of scattering mediated by a very light gauge boson. In SIDM models one has a "dark photon" (or "hidden-photon") instead of the usual photon but still if the mediator mass $m_\phi \ll m_{DM} v \Rightarrow \sigma_{self} \sim v^{-4}$. Prove that if $\sigma_{self} \simeq \sigma_{max}$ in dwarfs, then σ_{self} would be $\ll \sigma_{max}$ in clusters (so SIDM becomes just CDM at large scales).

Exercise 3 - Lower limit on the dark matter (DM) particle mass due to phase space arguments.

3.1 Assume the DM particle is a boson. Bosons tend to occupy the same lowest energy state. The occupation number is so high, that the DM behaves as a classical field obeying a wave equation (see Hu, Barkana and Gruzinov, astro-ph/0003365 P. R. L. 85 (2000) 1158). The problem at hand becomes formally the same as that of a particle of mass equal to the DM mass, m, in a potential well of the size of the dark halo. So we can use the Pauli's uncertainty principle $\Delta x \, \Delta p \gtrsim 1$ with $\Delta x \simeq 2 R_{halo}$ and $\Delta p \simeq m v$. Here v is the characteristic speed you estimated in Excercise 1. The most stringent lower limit on m from this relation comes from dwarf galaxies (do you see why?). Using the data and results of Excercise 1 prove that the limit obtained from dwarf galaxies is $m \gtrsim 10^{-22}$ eV.

[A DM particle with $m \simeq 10^{-22}$ eV was called "fuzzy DM" by Hu, Barkana and Gruzinov in the paper mentioned above. It is DM in a cold Bose-Einstein condensate, similar to axion DM.]

Exercise 3 (continuation)

3.2 Now let us assume that the DM particle is a fermion. The argument is different than for a boson due to Pauli's Exclusion Principle. For a fermion the phase space density is always $f(x,p) < 1$, so

$$M_{halo} = m \int f(x,p)\, d^3x\, d^3p \lesssim m\left(\frac{4}{3}\pi R_{halo}^3\right) \int d^3p$$

and for our estimate we can use $\int d^3p \simeq \Delta p^3$ and $\Delta p \simeq m v$ (you may see the original paper of Tremaine and Gunn PRL 42 (1979) 407 to get a more complete explanation. You may see also later papers by Madsen PRD 44 (1991) 999 and Horiuchi et al. PRD 89 (2014) 025017, 1311.0283).

Again dwarf galaxies provide the best lower limit. Prove that this limit is $m >$ few eV.

(this is the so called "Tremaine and Gunn limit").

Exercise 4 - Flux of dark matter (DM) particles on Earth

4.1 Define flux, i.e. number of particles traversing a surface per unit area per unit time, in terms of n = number density = number of particles per unit volume, and v = average speed particles.

In the following you may use for v either the characteristic speed for DM particles in the dark halo of the Milky Way (that you found in Exercise 1) or the speed of the Sun around the galaxy $v_\odot \simeq 220\,\mathrm{km/s}$ (or you may just use $v_\odot \simeq 10^{-3} c$ or just $v_\odot \simeq 10^{-3}$ in natural units). Also, we will use $\rho = 0.3\ \frac{\mathrm{GeV}}{\mathrm{cm}^3}$ as the local energy density of the DM ("local" meaning at the position of the solar system in our galaxy).

4.2 Find the characteristic flux of dark matter particles in units of number of particles per$(\mathrm{cm}^2 \mathrm{s})$ as function of the DM particles mass m.

4.3 How many DM particles (again, given as function of m) are on average in a 1 liter soda bottle?

Exercise 5 WIMPs interact coherently with nuclei

Consider the elastic collision of dark matter (DM) particles of mass m with a target of mass M_T.

5.1 Show that for $m \gtrsim$ GeV the typical momentum tranfer q (momentum imparted to the target, initially at rest, so that the target recoil energy is $E_R = q^2/2M_T$) is such that the interaction is coherent when the target is a nucleus.

Help. The radius of a nucleus is $R_N \simeq 1.25$ fm $A^{1/3}$ where A is the mass number A, $M_T \simeq A$ GeV. You may use the limits $m \ll M_T$ and $m \gg M_T$ to establish typical values of q.

5.2 Considering that present direct DM detection experiments have at present energy thresholds not lower than a fraction of keV, show that the maximum energy deposited in an elastic collision by a Light Dark Matter (LDM) particle, defined as DM particles with 1 keV $\lesssim m \lesssim$ 100 MeV is below threshold for detection.

Help: the lightest nuclear mass in use in Direct DM detection is about $M_T \simeq 10$ GeV.

Exercise 6 The minimum WIMP speed v_{min}

An important parameter entering into the differential rate is v_{min}, the minimum speed a WIMP must have to impart a particular recoil energy E_R to a nucleus. The nucleus is initially at rest.

6.1 Prove that for elastic collisions v_{min} is

$$v_{min} = \sqrt{\frac{M E_R}{2\mu^2}}$$

Where M is the target nucleus mass, m is the WIMP mass and μ is the reduced mass $\frac{1}{\mu} = \frac{1}{M} + \frac{1}{m}$

6.2 Consider inelastic scattering instead, in which a dark matter particle χ of mass m scatters into another state χ' with mass $m' = m + \delta$. It is usually the case that $|\delta| << m$. In an endothermic scattering $\delta < 0$, and in an exothermic scattering $\delta > 0$.

With the assumption $|\delta| << m$, prove that

$$v_{min} = \left| \sqrt{\frac{M E_R}{2\mu^2}} + \frac{\delta}{\sqrt{2 M E_R}} \right|$$

Notice that for $\delta > 0$ only DM particles with high speeds, $v > v_{min}$, have enough energy to up-scatter. Because there are less DM particles with large v, the rate in targets of small M is smaller. Targets with large mass are favored. For $\delta < 0$, $v_{min} = 0$ for $E_\delta = \frac{\mu \delta}{M}$.
With $v_{min} = 0$ all DM particles can scatter (rate is higher) if $E_\delta > E_{threshold}$ which requires with small enough mass.

Exercise 7 The halo function $\eta(v_{min})$

The differential event rate in a direct detection experiment depends on the "halo integral" or "halo function"

$$\eta(v_{min}, t) = \int_{v_{min}}^{\infty} \frac{f(\vec{v}, t)}{v} d^3\vec{v}$$

where $\vec{v}$ is the velocity of WIMPs relative to the Earth ($\vec{v}$ is actually the velocity of a WIMP relative to the detector and in our formulas we are neglecting the small velocity of the detector relative to the center of the Earth). $v = |\vec{v}|$ and you computed v_{min} in Exercise 6 -

In the Standard Halo model, the dark matter velocity distribution in the Galactic rest frame is a Maxwell-Boltzmann distribution with a cut-off at the local escape speed from the Galaxy v_{esc}. Including v_{esc} complicates considerably the equations, so let us take for the moment $v_{esc} \to \infty$. With this assumption

$$f_{gal}(\vec{v}_{WG}) = \frac{1}{(\pi v_0^2)^{3/2}} e^{-(\vec{v}_{WG}^{\,2}/v_0^2)}$$

where v_0 is the local circular velocity. To transform the distribution function from the Galactic frame to the detector's frame we use a Galilean transformation writing $\vec{v}_{WG}$, the velocity of a WIMP with respect to

Exercise 7 (continuation)

the galaxy in terms of $\vec{v}_\odot$, the velocity of the Sun relative to the Galactic Rest Frame, and $\vec{v}_\oplus(t)$, the velocity of Earth relative to the Sun (which is a function of the time t due to the rotation of Earth around the sun:

$$\vec{v}_{WG} = \vec{v}_\odot + \vec{v}_\oplus(t) + \vec{v}$$

(Notice that in writing f_{gal} we have used that the average WIMP velocity in the Galactic Rest Frame is zero.)

Then, $\quad f(\vec{v}, t) = f_{gal}\left(\vec{v} + \vec{v}_\oplus(t) + \vec{v}_\odot\right)$

7.1 Prove that neglecting the motion of the Earth with respect to the Galaxy (i.e. setting $\vec{v}_\odot + \vec{v}_\oplus(t) = 0$) one obtains a simple expression for η,

$$\eta(v_{min}) = \frac{2}{\sqrt{\pi}\, v_0}\, e^{-(v_{min}^2/v_0^2)}$$

7.2 Show that calling $\vec{v}_E = \vec{v}_\odot + \vec{v}_\oplus(t)$, $v_E = |\vec{v}_E|$ the halo function η is

$$\eta(v_{min}) = \frac{1}{2 v_E}\left[\operatorname{erf}\left(\frac{v_{min} + v_E}{v_0}\right) - \operatorname{erf}\left(\frac{v_{min} - v_E}{v_0}\right)\right]$$

where erf is the error function $\operatorname{erf}(a) = \frac{2}{\sqrt{\pi}} \int_0^a dx\, e^{-x^2}$.

Exercise 7 (continuation)

7.3 Let us now consider a stream of dark matter particles, assuming a simple model in which all particles reach us with the same velocity $\vec{V}$. This is already the velocity relative to the laboratory (so $\vec{V}_{SG} = \vec{V} + \vec{v}_{\oplus}(t) + \vec{v}_{\odot} = \vec{V} + \vec{v}_E(t)$, where $\vec{V}_{SG}$ is the velocity of the stream in the Galactic Rest Frame).

Thus $$f(\vec{v}) = \delta(\vec{v} - \vec{V})$$

Find and plot $\eta(v_{min})$ for this stream.

Notice that η is a function of t because $\vec{V}$ is a function of t $\quad \vec{V}(t) = \vec{V}_{SG} - \vec{v}_E(t)$. Thus, qualitatively, prot $\eta(v_{min}, t)$ for two different times, one with larger v_E than the other (v_E changes by about 10%, it is maximum close to June 1st and minimum 6 months later)

Exercise 8 Estimate of the differential recoil rate

8.1 Estimate the differential rate of nuclear recoil events due to the collision of WIMPs in a Xe detector for the following parameters

- WIMP mass $m = 100$ GeV
- recoil energy $E = 10$ keV
- local dark matter density $\rho = 0.3$ GeV/cm^3
- assume the SHM with local circular speed $v_0 = 220$ km/s and for simplicity neglect Earth's velocity with respect to the galaxy ($\vec{v}_E = 0$)
- Assume a spin-independent interaction with equal WIMP coupling with neutrons and protons $f_n = f_p$ and assume coherent scattering off the entire nucleus, so $F^2(q) = 1$.
- Assume elastic scattering

Give the differential rate $\frac{dR}{dE_R}$ in units of $\frac{\text{events}}{\text{kg keV yr}}$

8.2 Same as in 8.1 but now assume inelastic scattering with $\delta = 100$ keV

8.3 By how much changes the rate in 8.1 if we make the same assumptions as in 8.1 but $\frac{f_n}{f_p} = -0.7$

Mass of Xe is $M = 131.3$ amu (atomic mass unit)
1 amu = 0.931 GeV. The atomic mass number is $A = 131$.