Exercise sheet I. Freeze-out Mechanism

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I.a.- Compute the hot dark matter abundance for a fermion with n degrees of freedom and mass m.

I.b.- Compute the annihilation cross section into the total bosonic and fermionic content of the SM of a Dirac fermion with mass m and hypercharge Y. Assume that the mass is larger than the electroweak symmetry breaking scale.

I.c.- Estimate the cold dark matter abundance for such a Dirac fermion.

THERMAL FREEZE-OUT ABUNDANCES

In order to calculate the thermal relic abundance, we will use the standard techniques given in [1, 2] in two limiting cases, either relativistic (hot) or non-relativistic (cold) at decoupling. In this section we will review the basic steps of the calculation method.

The evolution of the number density n_α of a stable particle α, interacting with SM particles in an expanding universe is given by the Boltzmann equation:

$$\frac{dn_\alpha}{dt} = -3Hn_\alpha - \langle\sigma_A v\rangle(n_\alpha^2 - (n_\alpha^{eq})^2) \tag{1}$$

where

$$\sigma_A = \sum_X \sigma(\pi^\alpha\pi^\alpha \to X) \tag{2}$$

is the total annihilation cross section of annihilation of α particles into SM particles X summed over final states. The $-3Hn_\alpha$ term, with H the Hubble parameter, takes into account the dilution of the number density due to the universe expansion. These are the only terms which could change the number density of α particles. In fact, since they are stable they do not decay into other particles.

The thermal average $\langle\sigma_A v\rangle$ of the total annihilation cross section times the relative velocity is given by:

$$\langle\sigma_A v\rangle = \frac{1}{n_{eq}^2}\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3} f(E_1)f(E_2)\frac{w(s)}{E_1E_2}\,, \tag{3}$$

where:

$$w(s) = E_1E_2\sigma_A v_{rel} = \frac{s\sigma_A}{2}\sqrt{1-\frac{4M^2}{s}}\,, \tag{4}$$

with M the mass of the α particle. The Mandelstam variable s can be written in terms of the components of the four momenta of the two α particles p_1 and p_2 as $s = (p_1+p_2)^2 = 2(M^2 + E_1E_2 - |\vec{p}_1||\vec{p}_2|\cos\theta)$. Assuming vanishing chemical potential, the α distribution functions are:

$$f(E) = \frac{1}{e^{E/T}+a} \tag{5}$$

with $a=0$ for Maxwell-Boltzmann, $a=1$ for Fermi-Dirac, and $a=-1$ for Bose-Einstein. In the case of non-relativistic relics $T \ll 3M$, the Maxwell-Boltzmann distribution is a good approximation and we will use it for simplicity for cold relics. Finally, the equilibrium abundance is given by:

$$n_{eq} = \int \frac{d^3p}{(2\pi)^3} f(E) \tag{6}$$

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From (??), the thermal average will include, to leading order, annihilations into all the SM particle-antiparticle pairs. If the universe temperature is above the QCD phase transition ($T > T_c$), we consider annihilations into quark-antiquark and gluons pairs. If $T < T_c$ we include annihilations into light hadrons. For the sake of definiteness we will take a critical temperature $T_c \simeq 170$ MeV, although the final results are not very sensitive to the concrete value of this parameter.

In order to solve the Boltzmann equation we introduce the new variables: $x = M/T$ and $Y = n/s$ with s the universe entropy density. We will assume that the total entropy of the universe is conserved, i.e. $S = a^3 s = \text{const}$, where a is the scale factor of the universe and we will make use of the Friedmann equation:

$$H^2 = \frac{8\pi}{3M_P^2}\rho \tag{7}$$

where the energy density in a radiation dominated universe is given by:

$$\rho = g_{eff}(T)\frac{\pi^2}{30}T^4 \tag{8}$$

In a similar way, the entropy density reads:

$$s = h_{eff}(T)\frac{2\pi^2}{45}T^3 \tag{9}$$

where $g_{eff}(T)$ and $h_{eff}(T)$ denote the effective number of relativistic degrees of freedom contributing to the energy density and the entropy density respectively at temperature T (T being the temperature of the photon background). Notice that for $T >$ MeV we have $h_{eff} \simeq g_{eff}$. Using these expressions we get:

$$\frac{dY}{dx} = -\left(\frac{\pi M_P^2}{45}\right)^{1/2}\frac{h_{eff}M}{g_{eff}^{1/2}x^2}\langle\sigma_A v\rangle(Y^2 - Y_{eq}^2) \tag{10}$$

where we have ignored the possible derivative terms dh_{eff}/dT.

The qualitative behaviour of the solution of this equation goes as follows: if the annihiliation rate defined as $\Gamma_A = n_{eq}\langle\sigma_A v\rangle$ is larger than the expansion rate of the universe H at a given x, then $Y(x) \simeq Y_{eq}(x)$, i.e., the branon abundance follows the equilibrium abundances. However, since Γ_A decreases with the temperature, it eventually becomes similar to H at some point $x = x_f$. From that time on branons are decoupled from the rest of matter or radiation in the universe and its abundance remains frozen, i.e. $Y(x) \simeq Y_{eq}(x_f)$ for $x \geq x_f$. For instance, for relativistic (hot) scalar particles, the equilibrium abundance reads:

$$Y_{eq}(x) = \frac{45\zeta(3)}{2\pi^4}\frac{1}{h_{eff}(x)}, \qquad (x \ll 3) \tag{11}$$

whereas for cold relics:

$$Y_{eq}(x) = \frac{45}{2\pi^4}\left(\frac{\pi}{8}\right)^{1/2}x^{3/2}\frac{1}{h_{eff}(x)}\,e^{-x}, \qquad (x \gg 3) \tag{12}$$

We see that for hot relics the equilibrium abundance is not very sensitive to the value of x. In the case of cold relics however, Y_{eq} decreases exponentially with the temperature, which implies that the sooner the decoupling occurs the larger the abundance.

Let us first consider the simple case of hot relics. Since its equilibrium abundance depends on x_f only through $h_{eff}(x_f)$, the relic abundance is not very sensitive to the exact time of decoupling. In this case, in order to calculate the decoupling temperature $T_f = M/x_f$, it is a good approximation to use the condition $\Gamma_A = H$. From the explicit expression of the Hubble parameter in a radiation dominated universe we have:

$$H(T_f) = 1.67\, g_{eff}^{1/2}(T_f)\frac{T_f^2}{M_P} = \Gamma_A(T_f) \tag{13}$$

which can be solved explicitly for T_f, expanding $\Gamma_A(T_f)$ for $T_f \gg M/3$. Once we know x_f, the relic abundance today ($Y_\infty \simeq Y(x_f)$) is given by (11). From this expression we can obtain the current number density of hot scalar relics and the corresponding energy density which is given by:

$$\Omega_{Br}h^2 = 7.83\cdot 10^{-2}\frac{1}{h_{eff}(x_f)}\frac{M}{\text{eV}} \tag{14}$$

The calculation of the decoupling temperature in the case of cold relics is more involved. The well-known result is given by:

$$x_f = \ln\left(\frac{0.038\, c\,(c+2) M_P M \langle\sigma_A v\rangle}{g_{eff}^{1/2}\, x_f^{1/2}}\right) \tag{15}$$

where $c \simeq 0.5$ is obtained from the numerical solution of the Boltzmann equation. This equation can be solved iteratively. The corresponding energy fraction reads:

$$\Omega_{Br} h^2 = 8.77 \cdot 10^{-11} \text{GeV}^{-2} \frac{x_f}{g_{eff}^{1/2}} \left(\sum_{n=0}^{\infty} \frac{c_n}{n+1} x_f^{-n}\right)^{-1} \tag{16}$$

where we have expanded $\langle\sigma_A v\rangle$ in powers of x^{-1} as:

$$\langle\sigma_A v\rangle = \sum_{n=0}^{\infty} c_n x^{-n} \tag{17}$$

Notice that in general, $Y_\infty \propto 1/\langle\sigma_A v\rangle$, i.e. the weaker the cross section the larger the relic abundance. This is the expected result, since, as commented before the sooner the decoupling occurs, the larger the relic abundance, and decoupling occurs earlier as we decrease the cross section. Therefore the cosmological bounds work in the opposite way as compared to those coming from colliders. Thus, a bound such as $\Omega_{Br} < \mathcal{O}(1)$ translates into a lower limit for the cross sections and not into an upper limit as those obtained from non observation in colliders.

[1] E.W. Kolb and M.S. Turner, *The Early universe* (Addison-Wesley, 1990).

[2] M. Srednicki, R. Watkins and K.A. Olive, *Nucl. Phys.* **B310**, 693 (1988); P. Gondolo and G. Gelmini, *Nucl. Phys.* **B360**, 145 (1991).