DARK MATTER

Problem Sheet 1: Classical evidence

1. **Beyond spherical symmetry**

   (a) Suppose that we observe a galaxy with a rotating gaseous disc that has rotational speed: $v_{\rm obs}(R) = v_0 = $ constant, where R is the radius in the disc plane. A reasonable model for the disc potential is called the **Miyamoto-Nagai potential**:

$$\Phi_*(R,z) = -\frac{GM_*}{\sqrt{R^2 + (a + \sqrt{b^2+z^2})^2}} \tag{1}$$

   where G is Newton's Gravitational constant; M_* is the mass of the stellar disc; and a and b set the scale height and length of the disc, respectively. It is very useful because it approximates real discs in the Universe, but has an *analytic* solution to the Poisson equation such that the density of the disc can be simply written down:

$$\rho_*(R,z) = \left(\frac{b^2 M}{4\pi}\right) \frac{aR^2 + (a + 3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})^2}{\left[R^2 + (a+\sqrt{z^2+b^2})^2\right]^{5/2} (z^2+b^2)^{3/2}} \tag{2}$$

   i. Using the above potential Φ_*, show that $v_*(R)$ for gas moving on circular orbits in the disc plane ($z = 0$) is given by:

$$v_*^2 = \frac{GM_* R^2}{\left[R^2 + (a+b)^2\right]^{3/2}} \tag{3}$$

   ii. What happens in the limit $R \to \infty$ to the disc density ρ_* in the disc plane ($z = 0$)? What happens to $v_*(R)$ in the same limit?

   iii. How does the above compare with the observation $v_* = v_0 = $ constant. What does this mean?

   (b) In the notes, we assumed that galaxies are spherical from which we derived:

$$v_*^2 = \frac{GM_*(R)}{R} \tag{4}$$

   For the same disc mass M_*, sketch how equation 4 compares to equation 3 (you can assume without loss of generality that $a+b = 1$). What does this tell you about the assumption of spherical symmetry?

   (c) Consider now the potential:

$$\Phi(R,z) = \frac{v_0^2}{2} \ln\left(R_c^2 + R^2 + \frac{z^2}{q^2}\right) \tag{5}$$

   where R_c and q are constants.

   i. What do the constants R_c and q mean (i.e. how does changing them change the potential?)

   ii. Calculate $v_c(R)$ for this potential. How does it compare with equation 3? Do you think this is a better model of real galaxies in the Universe?

   iii. From equation 4, in order to keep $v_c = $ const. we must increase the mass as $M(R) \propto R$. What do you think this tells us about the Logarithmic potential in equation 5?

2. **What dark matter is not**

   (a) Assume, as in the notes, that the Milky Way halo density profile is a spherical power law:

$$\rho_{\rm dm} = \rho_0 \left(\frac{r}{r_s}\right)^{-\alpha} \tag{6}$$

   where we have normalised the distribution so that the local dark matter density at the Solar neighbourhood $r = r_s \sim 8\,$kpc is given by ρ_0.

   Show that the Milky Way's rotation curve (the rotation speed of gas moving on a circular orbit) is given by:

$$v_c^2 = \frac{GM_{\rm dm}(r_s)}{r_s} + v_{c,b}^2(r_s) \tag{7}$$

where $v_{c,b}^2(r_s) \sim (150\,\mathrm{km/s})^2$ is the baryonic contribution to the rotation curve at r_s.

(b) Show that the enclosed dark matter mass $M_{\rm dm}(r_s)$ is given by:

$$M_{\rm dm}(r_s) = \frac{4\pi\rho_0 r_s^3}{3-\alpha} \tag{8}$$

(c) Calculate the dark matter density at $r_s \sim 8\,\mathrm{kpc}$, assuming $\alpha = 1$. Compare this with the total matter density at r_s, $\rho_{\rm tot} \sim 0.09\,\mathrm{M_\odot\,pc^{-3}}$. Is the ratio of the two surprising? What do you think it means?