X-ray Transition Radiation

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Abstract

X-ray transition radiation produced by a relativistic charge crossing the interface between media is considered in terms of the energy loss against additional electric field induced in the vicinity of the interface. X-ray transition radiation in a plate, in a stack of plates, and in complex radiators consisting more than two media are considered in details. The comparison of GEANT4 simulation with experimental data is discussed.

1 X-ray Transition Radiation Energy Loss

X-ray transition radiation (XTR) is widely used in experimental high energy physics for particle identification, especially for the selection of electrons in an environment of high hadron background. The standard theory of XTR describes the flux of XTR photons far from the XTR radiator in the so called wave zone. This is not the case when the XTR detector is placed directly inside the radiator (see, for example, the design of the XTR tracker of the ATLAS experiment). For such a radiator geometry including more than two media other versions of the XTR theory may be applied, e.g., those based on the calculation of XTR energy loss. Our main goal is to describe the XTR energy loss produced by a relativistic charged particle crossing different geometries of the media interfaces against additional electric field induced near the interfaces. The XTR energy loss provides the spectrum and angular distribution of XTR photons emitted in the vicinity of the particle trajectory. We will describe the X-ray transition radiation energy loss taking into account the absorption of XTR photons in all media involved.

Let a charge e, moving with the constant velocity v along the direction of unit vector $\mathbf{n}$ ($\mathbf{v} = v\mathbf{n}$), cross the interface between two media with dielectric permittivities ϵ_1 and ϵ_2, respectively. The part of the total mean work done by the charge against the field produced by itself resulting in the emission of XTR photons, $\bar{\Delta}_\perp^{(12)}$, can be expressed as:

$$\bar{\Delta}_\perp^{(12)} = -e\int_{-\infty}^{\infty} \mathbf{E}_\perp(\mathbf{v}t, t)\mathbf{v}dt = \delta W_{12} + U_2,$$

where $\mathbf{E}_\perp(\mathbf{v}t, t)$ is the additional electric field produced by the charge in its current position $\mathbf{r} = \mathbf{v}t$ near interface, δW_{12} is the energy of macroscopic mass renormalisation when the charge crosses the interface from the first medium to the second one (it reflects the fact that a charged particle in a medium possesses a different mass from that in vacuum), and U_2 is the field energy of XTR photons in the second medium. The index $\perp$ means that we consider the part of the electric field perpendicular to the direction $\mathbf{k}$ of the emitted XTR photon, so that relation describes the energy loss due to transverse electro-magnetic excitations of a medium (photons in a medium).

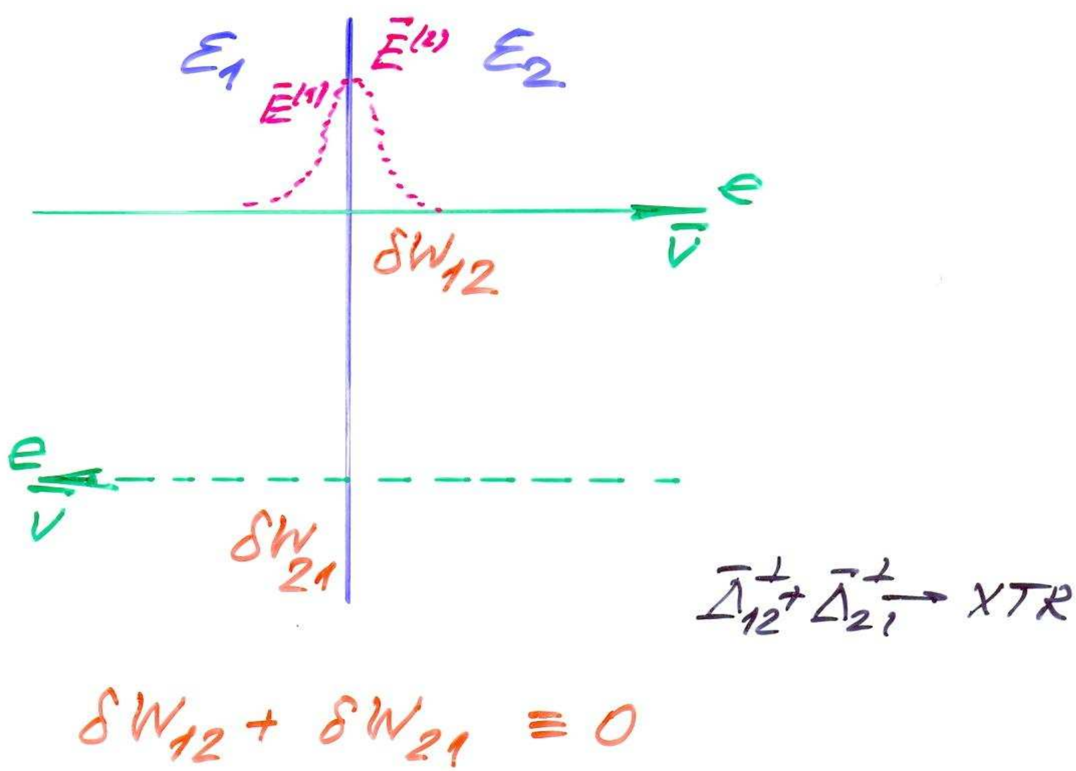

Figure 1: Diagram of a charged particle crossing the single interface between two absorbing media.

We will limit our consideration to the ultra-relativistic case ($v \sim c$, where c is the speed of light in vacuum), when XTR is emitted forward (to the second medium) at small angles ($\sim 1/\gamma$, where $\gamma \gg 1$ is the particle Lorentz factor) around the direction of the charge motion $\mathbf{n}$ ($\mathbf{v} = v\mathbf{n}$). To describe the energy loss in terms of the field energy of XTR photons, U, we have to reduce the energy of the macroscopic mass renormalisation. To do that we consider the energy loss when the charge crosses the intersection between the second and the first media moving with the same velocity in opposited direction:

$$\bar{\Delta}_{\perp}^{(21)} = \delta W_{21} + U_1.$$

It is obviously that $\delta W_{12} + \delta W_{21} = 0$ and in the X-ray region $U_1 \sim U_2 \sim U$, since $\epsilon_1 \sim \epsilon_2 \sim 1$. As a result, we obtain:

$$U = \frac{\bar{\Delta}_{\perp}^{(12)} + \bar{\Delta}_{\perp}^{(21)}}{2}.$$

For the definition of the energy-angle distribution of the XTR photons we expand the electric field in the Fourier integral (no 2π factors!):

$$\mathbf{E}_{\perp}(\mathbf{r}, t) = \iint d\omega\, d\mathbf{k}\, \mathbf{E}_{\perp}(\mathbf{k}, \omega) \exp\left[i(\mathbf{k}\mathbf{r} - \omega t)\right], \quad \mathbf{E}_{\perp}(\mathbf{k}, \omega)\mathbf{k} = 0,$$

where $\mathbf{k}$ and ω are the wave vector and the frequency of the XTR photon, respectively. Then relation for XTR energy loss will be:

$$\bar{\Delta}_{\perp}^{(12)} = -ev\int_{-\infty}^{\infty} d\omega \int d\mathbf{k} E_{n\,\perp}^{(12)}(\mathbf{k},\omega) \int_{-\infty}^{\infty} dt \exp\left[it(\mathbf{kv}-\omega)\right] =$$

$$= -2\pi ev\int_{-\infty}^{\infty} d\omega \int d\mathbf{k} E_{n\,\perp}^{(12)}(\mathbf{k},\omega)\delta(\omega-\mathbf{kv}),$$

where $E_{n\,\perp}^{(12)}(\mathbf{k},\omega)$ is the component of $\mathbf{E}_{\perp}(\mathbf{k},\omega)$ parallel to the direction of the charge motion $\mathbf{n}$, and δ is the Dirac delta function. The latter will be used to find the integral with respect to the module of the wave vector k:

$$\int d\mathbf{k} = 2\pi\int_{0}^{\infty} k^2 dk \left\{\int_{-1}^{0} d\cos\theta + \int_{0}^{1} d\cos\theta\right\},$$

where θ is the angle between $\mathbf{n}$ and $\mathbf{k}$. One can see that the delta function connects the signs of ω and $\cos\theta$. Therefore the integrals with respect to ω and $\cos\theta$ are divided into two regions where both ω and $\cos\theta$ have the same sign.

After reducing the integration with respect to the frequency to the physical region, $\omega \geq 0$ and with respect to $\cos\theta$ over the forward hemisphere, relation for XTR-dEdx becomes:

$$\bar{\Delta}_{\perp}^{(12)} = -4\pi^2 e\, v \int_0^{\infty} d\omega \int_0^1 d\cos\theta \int_0^{\infty} k^2 d\,k \left\{ E_{n\,\perp}^{(12)}(\mathbf{k},\omega)\delta(\omega - \mathbf{k}\mathbf{v}) + \right.$$

$$\left. + E_{n\,\perp}^{(12)}(\mathbf{k},-\mathbf{k}\mathbf{v},-\omega)\delta(-\omega + \mathbf{k}\mathbf{v}) \right\}.$$

In the X-ray region the main contribution to the integral with respect to $\cos\theta$ will be from $\theta \sim 0$. Therefore we can make the substitution $2(1-\cos\theta) \sim \theta^2$ and change the upper limit for the integration with respect to θ^2 to be infinity, since the main contribution to the integral will be from the region of $\theta \ll 1$. In addition, the integration with respect to k results in $k = \omega/v\cos\theta \sim \omega/c$ and appearance of a factor from the delta function, $1/v\cos\theta \sim 1/v$. Therefore, the energy-angle distribution of the energy loss will be:

$$\frac{d^2\bar{\Delta}_{\perp}^{(12)}}{\hbar d\omega\, d\theta^2} = -\frac{2\pi^2 e}{\hbar}\left(\frac{\omega}{c}\right)^2 \left\{ E_{n\,\perp}^{(12)}(\mathbf{k},\omega) + E_{n\,\perp}^{(12)}(-\mathbf{k},-\omega) \right\}, \quad \mathbf{k}\mathbf{v} = \omega.$$

Here we introduced the XTR photon energy, $\hbar\omega$, where $\hbar$ is Planck's constant.

It is obvious that $E_{n\perp}^{(12)}(-\mathbf{k},-\omega) = E_{n\perp}^{(12)*}(\mathbf{k},\omega)$, where $*$ means the complex conjugation operation. The energy-angle distribution of the emitted energy of the XTR photons is then:

$$\frac{d^2U}{\hbar d\omega\, d\theta^2} = -\frac{2\pi^2 e}{\hbar}\left(\frac{\omega}{c}\right)^2 \mathrm{Re}\left\{E_{n\,\perp}^{(12)}(\mathbf{k},\omega) + E_{n\,\perp}^{(21)}(\mathbf{k},\omega)\right\}.$$

Averaging over the forward and backward directions of motion is needed for an odd number of interfaces of particular type only (12,21,12). If the number of interfaces is even, as, for example, for a number of plates of the second medium placed in different positions in the first medium, it is enough to consider the XTR energy loss for the forward direction of motion only. In particular, for one plate the energy-angular distribution of the emitted energy of the XTR photons reads (12,21):

$$\frac{d^2U}{\hbar d\omega\, d\theta^2} = -\frac{4\pi^2 e}{\hbar}\left(\frac{\omega}{c}\right)^2 \mathrm{Re}\left\{E_{n\,\perp}^{(121)}(\mathbf{k},\omega)\right\}, \quad \times\frac{1}{(2\pi)^4}.$$

The same results follow from our general consideration:

$$\bar{\Delta}_{\perp} = \frac{2}{1}\int_0^{\infty} d\omega \int_{K_3} d\mathbf{k}\, \mathrm{Re}\{\mathbf{j}^*(\mathbf{k},\omega)\cdot\mathbf{E}_{\perp}(\mathbf{k},\omega)\}, \quad \times(2\pi)^4.$$

For uniform motion with constant velocity $\mathbf{v} = v\mathbf{n}$ the current density reads:

$$\mathbf{j}(\mathbf{k},\omega) = 2\pi e\mathbf{v}\delta(\omega - \mathbf{k}\cdot\mathbf{v}), \quad d\mathbf{k} = 2\pi k^2 dk\, \sin\theta\, d\theta = \pi k^2 dk\, d\theta^2,$$

$$\delta(...) \to \frac{1}{v\cos\theta} \sim \frac{1}{v}, \quad k^2 \to \frac{\omega^2}{v^2\cos^2\theta} \sim \frac{\omega^2}{c^2}.$$

Therefore we have for the energy-angular distribution of the emitted energy of the XTR photons:

$$\frac{d^2\bar{\Delta}_{\perp}}{\hbar d\omega\, d\theta^2} = -\frac{4\pi^2 e}{\hbar}\left(\frac{\omega}{c}\right)^2 \mathrm{Re}\,\{E_{n\,\perp}(\mathbf{k},\omega)\}.$$

These relations will be used in the next section for the calculation of the XTR energy loss for different geometries of the medium interfaces.

2 Single Interface between Media

For simplicity, we consider the case when the first and second media are located in semi spaces $z < 0$ and $z > 0$, respectively, and the charge moves along the z axis ($\mathbf{r}(t) = \mathbf{v}t = \{0, 0, vt\}$). We start with equation for electric field in space time domain (using ϵ, $\epsilon_\perp$ representation):

$$\text{(Ampere-Maxwell):}\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{j},$$

$$\text{(Gauss):}\ \nabla \cdot \mathbf{D} = 4\pi\rho, \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\ \text{:(Faradey)},$$

with $\mathbf{D}_\perp = \hat{\epsilon}_\perp \mathbf{E}_\perp$ and $\mathbf{D}_\parallel = \hat{\epsilon}\mathbf{E}_\parallel$. We limit our consideration by the case of high frequencies $\hbar\omega > I_K$, where I_K is the binding energy of K-shell. In this frequency region $\epsilon_\perp \sim \epsilon = 1 - \omega_p^2/\omega^2$. We get the time derivative of Ampere-Maxwell law:

$$\nabla \times \frac{\partial \mathbf{B}}{\partial t} - \frac{1}{c}\frac{\partial^2 \mathbf{D}}{\partial^2 t} = \frac{4\pi}{c}\frac{\partial \mathbf{j}}{\partial t},$$

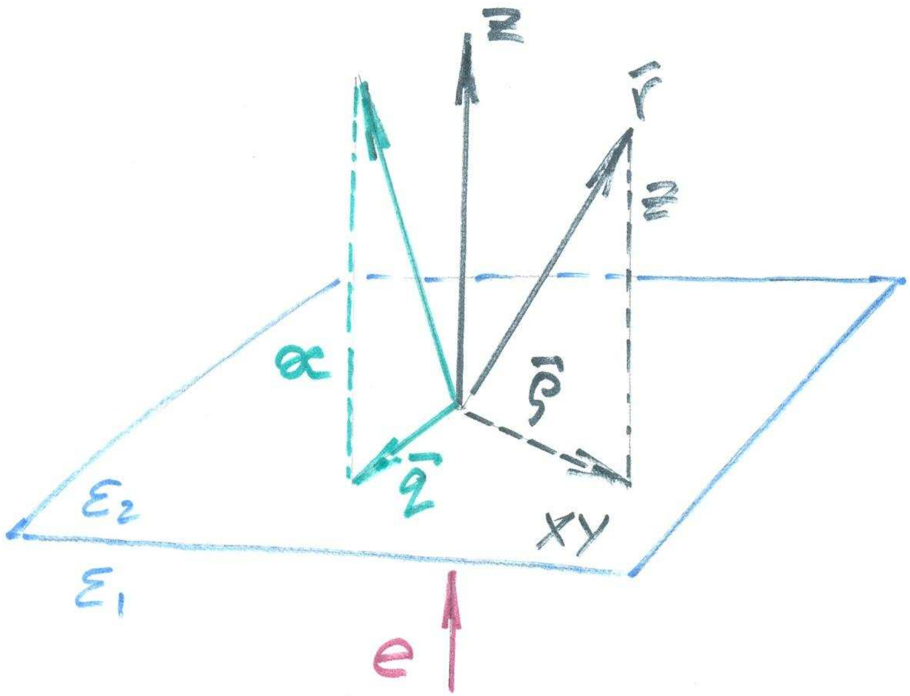

Figure 2: Some notations for the treatment of single interface between media, $\mathbf{r} = (\boldsymbol{\rho}, z)$ and $\mathbf{k} = (\mathbf{q}, \varkappa)$.

$$-c\nabla \times (\nabla \times \mathbf{E}) - \frac{1}{c}\frac{\partial^2 \mathbf{D}}{\partial^2 t} = \frac{4\pi}{c}\frac{\partial \mathbf{j}}{\partial t}, \quad \nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \Delta \mathbf{E},$$

$$-\nabla(\nabla \cdot \mathbf{E}) + \Delta \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{D}}{\partial^2 t} = \frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t}.$$

The value $\nabla \cdot \mathbf{E}$ is defined from the Gauss law:

$$\nabla \cdot (\hat{\epsilon}\mathbf{E}) = \hat{\epsilon}\nabla \cdot \mathbf{E} + (\nabla\hat{\epsilon}) \cdot \mathbf{E} = 4\pi\rho,$$

$$\Delta \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \hat{\epsilon}\mathbf{E}}{\partial^2 t} = \frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t} + \nabla\left\{\hat{\epsilon}^{-1}\left[4\pi\rho - (\nabla\hat{\epsilon}) \cdot \mathbf{E}\right]\right\}.$$

Let us consider the case when $\hat{\epsilon}$ does not depend explicitly on $\mathbf{r}$ and t in semi-spaces $z < 0$ and $z > 0$, having however in there the different values ϵ_1 and ϵ_2, respectively.

$$\Delta \mathbf{E} - \frac{\hat{\epsilon}}{c^2}\frac{\partial^2 \mathbf{E}}{\partial^2 t} = \frac{4\pi}{c^2}\frac{\partial \mathbf{j}}{\partial t} + 4\pi\hat{\epsilon}^{-1}\nabla\rho.$$

Since the dielectric properties are different in $z < 0$ and $z > 0$, we will define firstly the intermediate Fourier components of the electric field:

$$\mathbf{E}(\mathbf{q}, z, \omega) = \int \frac{d\boldsymbol{\rho}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \mathbf{E}(\rho, z, t) \exp\left[-i(\boldsymbol{\rho}\mathbf{q} - \omega t)\right],$$

where $\boldsymbol{\rho}$ and $\mathbf{q}$ are the components of the vectors $\mathbf{r}$ and $\mathbf{k}$ in the xy-plane respectively ($\mathbf{r} = (\boldsymbol{\rho}, z)$ and $\mathbf{k} = (\mathbf{q}, \varkappa)$, where $\varkappa$ is the component of $\mathbf{k}$ along the z axis). The total (particle and induced) fields $\mathbf{E}^{(j)tot}(\mathbf{q}, z, \omega)$ satisfy by the following equations in each medium ($j = 1, 2$):

$$\frac{\partial^2 \mathbf{E}^{(j)tot}}{\partial^2 t} - q^2 \mathbf{E}^{(\mathbf{j})\mathbf{tot}} + \epsilon_j \frac{\omega^2}{c^2} \mathbf{E}^{(\mathbf{j})\mathbf{tot}} =$$

$$= -4\pi \frac{i\omega}{c^2} \mathbf{j}(\mathbf{q}, z, \omega) + \frac{4\pi}{\epsilon_j} \left\{ i\mathbf{q}\rho(\mathbf{q}, z, \omega), \frac{\partial \rho(\mathbf{q}, z, \omega)}{\partial z} \right\},$$

$$\rho(\mathbf{q}, z, \omega) = \int\int \frac{d\rho\, dt}{(2\pi)^3} e\delta(\rho)\delta(z - vt) \exp\left[i(\omega t - \rho \cdot \mathbf{q})\right] = \frac{e}{(2\pi)^3 v} \exp\left[i\frac{\omega}{v} z\right],$$

$$\mathbf{j}(\mathbf{q}, z, \omega) = \frac{e\mathbf{v}}{(2\pi)^3 v} \exp\left[i\frac{\omega}{v} z\right], \quad \frac{\partial \rho(\mathbf{q}, z, \omega)}{\partial z} = i\frac{\omega}{v} \rho(\mathbf{q}, z, \omega).$$

And the field equations finally read:

$$\frac{\partial^2 \mathbf{E}^{(j)tot}}{\partial z^2} + \zeta_j^2 \mathbf{E}^{(j)tot} = \mathbf{A}^{(j)} \exp\left(i\frac{\omega}{v}z\right),$$

where ($\beta = v/c$):

$$\zeta_j^2 = \epsilon_j \frac{\omega^2}{c^2} - q^2, \quad \mathbf{A}^{(j)} = \frac{ie}{2\pi^2 v \epsilon_j}\left\{\mathbf{q}, \frac{\omega}{v}(1 - \epsilon_j \beta^2)\right\},$$

The (just math) solution in both media can be represented as:

$$\mathbf{E}^{(1)tot} = \mathbf{E}^{(1)} \exp\left(-i\zeta_1 z\right) - \mathbf{B}^{(1)} \exp\left(i\frac{\omega}{v}z\right),$$

$$\mathbf{E}^{(2)tot} = \mathbf{E}^{(2)} \exp\left(i\zeta_2 z\right) - \mathbf{B}^{(2)} \exp\left(i\frac{\omega}{v}z\right),$$

it is the sum of general with $A = 0$ and particular (particle field) solutions, where

$$\mathbf{B}^{(j)} = \frac{\mathbf{A}^{(j)}}{\frac{\omega^2}{v^2} - \zeta_j^2}.$$

Here $\mathbf{E}^{(j)}$ are the amplitudes of the additional transverse electromagnetic radiation fields expanding from the plane $z = 0$. These amplitudes can be derived from the boundary conditions for the total electric fields $\mathbf{E}^{(j)tot}$:

$$\epsilon_1 E_n^{(1)tot} = \epsilon_2 E_n^{(2)tot}, \quad E_\tau^{(1)tot} = E_\tau^{(2)tot}, \quad \mathbf{q} = q\boldsymbol{\tau},$$

where $\boldsymbol{\tau}$ is the unit vector in the xy-plane, and transverse conditions for the transition radiation fields:

$$(\nabla \cdot \mathbf{E}^{(j)} = 0) \rightarrow: \ (\mathbf{q} - \zeta_1 \mathbf{n})\mathbf{E}^{(1)} = 0, \quad (\mathbf{q} + \zeta_2 \mathbf{n})\mathbf{E}^{(2)} = 0.$$

In the ultra-relativistic case, when $\gamma = (1 - \beta^2)^{-1/2} \gg 1$, the main part of the transition radiation is emitted in the X-ray region and within small $\theta \sim \gamma^{-1}$. This means that the main contribution to the emitted XTR energy will be from $E_n^{(2)}$. Note that the terms proportional to $\mathbf{B}^{(j)}$ will contribute to Cherenkov radiation mainly in the spectral ranges where $Re\{\epsilon_j\} > 1$, which is not the case for the X-ray region. We will use the standard (in XTR theory) representation of the dielectric permittivities:

$$\epsilon_j = 1 - \frac{\omega_j^2}{\omega^2} + i\frac{c}{\omega l_j},$$

where ω_j and l_j are the plasma frequency and the photon absorption length in the j-th medium, respectively. In the region of small emission angles, and taking into account that for straight trajectory $\omega = \mathbf{kv}$, we have:

$$\cos\theta \sim 1 - \frac{\theta^2}{2}, \quad \beta^2 \sim 1 - \gamma^{-2}, \quad k \sim \varkappa \sim \frac{\omega}{v} \sim \frac{\omega}{c}, \quad q \sim \frac{\omega}{c}\theta,$$

$$\frac{\omega^2}{v^2} - \zeta_j^2 \simeq \frac{\omega}{cZ_j}, \quad \zeta_j - \frac{\omega}{v} \simeq -\frac{1}{2Z_j},$$

$$\mathbf{A}^{(j)} \simeq A\left\{\frac{\omega}{c}\theta\boldsymbol{\tau}, \frac{1}{Z_j} - \frac{\omega}{c}\theta^2\right\}, \quad A = \frac{ie}{2\pi^2 c},$$

where we introduce the *complex formation zone*, Z_j, of X-ray transition radiation in the j-th medium:

$$Z_j = \frac{L_j}{1 - i\dfrac{L_j}{l_j}}.$$

In the case of a transparent medium $l = \infty$, the complex formation zone is reduced to the *coherence length* L_j of XTR:

$$L_j = \frac{c}{\omega}\left[\gamma^{-2} + \frac{\omega_j^2}{\omega^2} + \theta^2\right]^{-1}.$$

Note that $Z_j(-\omega) = -Z_j^*(\omega)$, since $Im\{\epsilon_j(-\omega)\} = -Im\{\epsilon_j(\omega)\}$. With this notation the transverse conditions will be transformed to:

$$E_\tau^{(1)} = \frac{E_n^{(1)}}{\theta}, \quad E_\tau^{(2)} = -\frac{E_n^{(2)}}{\theta},$$

and the boundary conditions will give:

$$n: \quad E_n^{(1)} + A\theta^2 Z_1 = E_n^{(2)} + A\theta^2 Z_2,$$

$$\tau: \quad E_n^{(1)} - A\theta^2 Z_1 = -E_n^{(2)} - A\theta^2 Z_2.$$

Subtraction of these relations results in:

$$E_n^{(2)}(\mathbf{q},\omega) = \frac{ie}{2\pi^2 c}\theta^2(Z_1 - Z_2), \text{ while, if summation, } E_n^{(1)} = 0.$$

The corresponding Fourier transformation $E_n^{(2)}(\mathbf{k},\omega)$ of $E_n^{(2)}(\mathbf{q},\omega)\exp(i\zeta_2 z)$ will be:

$$E_n^{(2)}(\mathbf{k},\omega) = \int_0^\infty \frac{dz}{2\pi} E_n^{(2)}(\mathbf{q},\omega)\exp\left[i(\zeta_2 - \varkappa)z\right] = \frac{e\theta^2}{2\pi^3 c} Z_2(Z_1 - Z_2).$$

The latter relation clearly shows that $E_n^{(2)}(\mathbf{k},\omega)$ depends on ω by Z only. Therefore we have:

$$E_n^{(2)}(-\mathbf{k},-\omega) = \frac{e\theta^2}{2\pi^3 c} Z_2^*(Z_1^* - Z_2^*) = E_n^{(2)*}(\mathbf{k},\omega).$$

The total work done by the charge against the radiation field $E_n^{(2)}(\mathbf{k},\omega)$ expanding to the second medium will be now:

$$\bar{\Delta}_\perp^{(12)} = -\frac{e^2}{\pi c}\int_0^\infty d\omega \int_0^\infty d\theta^2 \left(\frac{\omega}{c}\right)^2 \theta^2 Re\left\{Z_2(Z_1 - Z_2)\right\}.$$

Combining the latter equation with the similar expression for:

$$\bar{\Delta}_\perp^{(21)} = -\frac{e^2}{\pi c}\int_0^\infty d\omega \int_0^\infty d\theta^2 \left(\frac{\omega}{c}\right)^2 \theta^2 Re\left\{Z_1(Z_2 - Z_1)\right\}.$$

$$Z_2(Z_1 - Z_2) + Z_1(Z_2 - Z_1) = -(Z_1 - Z_2)^2,$$

one gets the final result for the energy-angle distribution of emitted XTR energy:

$$\frac{d^2U}{\hbar d\omega\, d\theta^2} = \frac{\alpha}{\pi}\left(\frac{\omega}{c}\right)^2 \theta^2 \mathrm{Re}\left\{(Z_1 - Z_2)^2\right\},$$

where α is the fine structure constant (we consider for simplicity the particle charge e to be equal to the electron charge). One can see that in absorbing media the emitted XTR energy is defined by the real part of the square of the difference of the medium complex formation zones. In transparent media this relation reduces to

$$\frac{d^2U}{\hbar d\omega\, d\theta^2} = \frac{\alpha}{\pi}\left(\frac{\omega}{c}\right)^2 \theta^2 (L_1 - L_2)^2 = \frac{\alpha}{\pi}\left(\frac{\omega}{c}\right)^2 \theta^2 |L_1 - L_2|^2,$$

in accordance with the standard theory. For the energy-angle distribution of the mean number of emitted XTR photons we have, by definition,

$$\frac{d^2\bar{N}}{\hbar d\omega\, d\theta^2} = \frac{1}{\hbar\omega}\frac{d^2U}{\hbar d\omega\, d\theta^2} = \frac{\alpha}{\pi\hbar c^2}\omega\theta^2 \mathrm{Re}\left\{(Z_1 - Z_2)^2\right\}.$$

3 Detector Window

Let us consider a relativistic charge crossing two interfaces between the first and second media ($z = 0$) and between the second and the third media ($z = a$), respectively. Such a configuration is realised, for example, when the charge enters an XTR gaseous detector (the third medium in this case will be the detector gas mixture). The electric fields will be the following:

$$\mathbf{E}^{(1)} \exp\left(-i\zeta_1 z\right) - \mathbf{B}^{(1)} \exp\left(i\frac{\omega}{v}z\right),$$

$$\mathbf{E}^{(2)} \exp\left(i\zeta_2 z\right) + \mathbf{E}^{(3)} \exp\left(-i\zeta_2 z\right) - \mathbf{B}^{(2)} \exp\left(i\frac{\omega}{v}z\right),$$

$$\mathbf{E}^{(4)} \exp\left(i\zeta_3 z\right) - \mathbf{B}^{(3)} \exp\left(i\frac{\omega}{v}z\right),$$

in the first, second and third media, respectively.

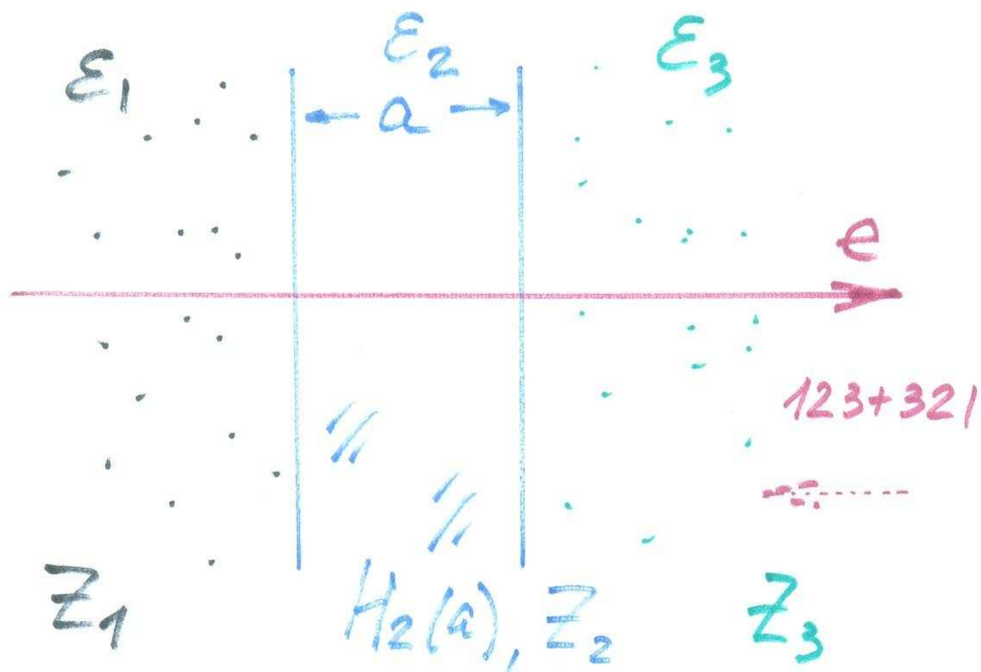

Figure 3: Diagram of a charged particle crossing a detector window 1-2-3.

Note that, since the second medium is limited from both sides, the radiation fields can expand there in both directions along the z-axis. The transverse conditions will be:

$$E_\tau^{(2k-1)} = \frac{E_n^{(2k-1)}}{\theta}, \quad E_\tau^{(2k)} = -\frac{E_n^{(2k)}}{\theta}, \quad k = 1, 2.$$

The boundary conditions at the $z = 0$ read:

$$n: \quad E_n^{(1)} + A\theta^2 Z_1 = E_n^{(2)} + E_n^{(3)} + A\theta^2 Z_2,$$

$$\tau: \quad E_n^{(1)} - A\theta^2 Z_1 = -E_n^{(2)} + E_n^{(3)} - A\theta^2 Z_2.$$

From these equations we have again for $E_n^{(2)}$:

$$E_n^{(2)}(\mathbf{q}, \omega) = A\theta^2 (Z_1 - Z_2), \quad (A = \frac{ie}{2\pi^2 c}).$$

The boundary conditions at $z = a$ read:

$$n: \quad E_n^{(3)} \exp(-i\zeta_2 a) + E_n^{(2)} \exp(i\zeta_2 a) + A\theta^2 Z_2 \exp\left(i\frac{\omega}{v}a\right) =$$

$$= E_n^{(4)} \exp(i\zeta_3 a) + A\theta^2 Z_3 \exp\left(i\frac{\omega}{v}a\right),$$

$$\tau : \quad E_n^{(3)} \exp(-i\zeta_2 a) - E_n^{(2)} \exp(i\zeta_2 a) - A\theta^2 Z_2 \exp\left(i\frac{\omega}{v}a\right) =$$

$$= -E_n^{(4)} \exp(i\zeta_3 a) - A\theta^2 Z_3 \exp\left(i\frac{\omega}{v}a\right).$$

The subtraction of these relations results in $E_n^{(4)}$ being given by:

$$E_n^{(4)}(\mathbf{q},\omega) = A\theta^2(Z_1 - Z_2)H_2(a)H_3(-a) + A\theta^2(Z_2 - Z_3)H_3(-a),$$

where we introduce the convenient function $H_j(z)$:

$$H_j(z) = \exp\left[i\left(\zeta_j - \frac{\omega}{v}\right)z\right] \simeq \exp\left[-\frac{iz}{2Z_j}\right] =$$

$$= \exp\left(-\frac{z}{2\lambda_j}\right)\left[\cos\left(\frac{z}{2L_j}\right) - i\sin\left(\frac{z}{2L_j}\right)\right],$$

with the following obvious properties:

$$H_j(z_1)H_j(z_2) = H_j(z_1 + z_2), \quad \exp\left[i\left(\zeta_j - \zeta_k\right)z\right] = H_j(z)H_k(-z).$$

The Fourier image of the n-component of the radiation fields, expanding along the z-axis, reads:

$$E_n^{(123)}(\mathbf{k},\omega) = \int_0^a \frac{dz}{2\pi} E_n^{(2)}(\mathbf{q},\omega) H_2(z) + \int_a^\infty \frac{dz}{2\pi} E_n^{(4)}(\mathbf{q},\omega) H_3(z) =$$

$$= \frac{2A\theta^2}{2\pi i} \left\{ Z_2(Z_1 - Z_2)\left[1 - H_2(a)\right] + Z_3(Z_1 - Z_2)H_2(a) + Z_3(Z_2 - Z_3) \right\}.$$

Taking half of the sum of the latter relation with $E_n^{(321)}(\mathbf{k},\omega)$, with the substitution $1 \leftrightarrows 3$ one gets, for the energy-angular distribution of the emitted XTR energy,

$$\frac{d^2U^{(123)}}{\hbar d\omega\, d\theta^2} = \frac{\alpha}{\pi}\left(\frac{\omega}{c}\right)^2 \theta^2 \mathrm{Re}\left\{ R^{(123)} \right\},$$

where the factor $R^{(123)}$ reflects the interference of contributions from different interfaces between media:

$$R^{(123)} = (Z_1 - Z_2)^2 + (Z_2 - Z_3)^2 + 2(Z_1 - Z_2)(Z_2 - Z_3)H_2(a).$$

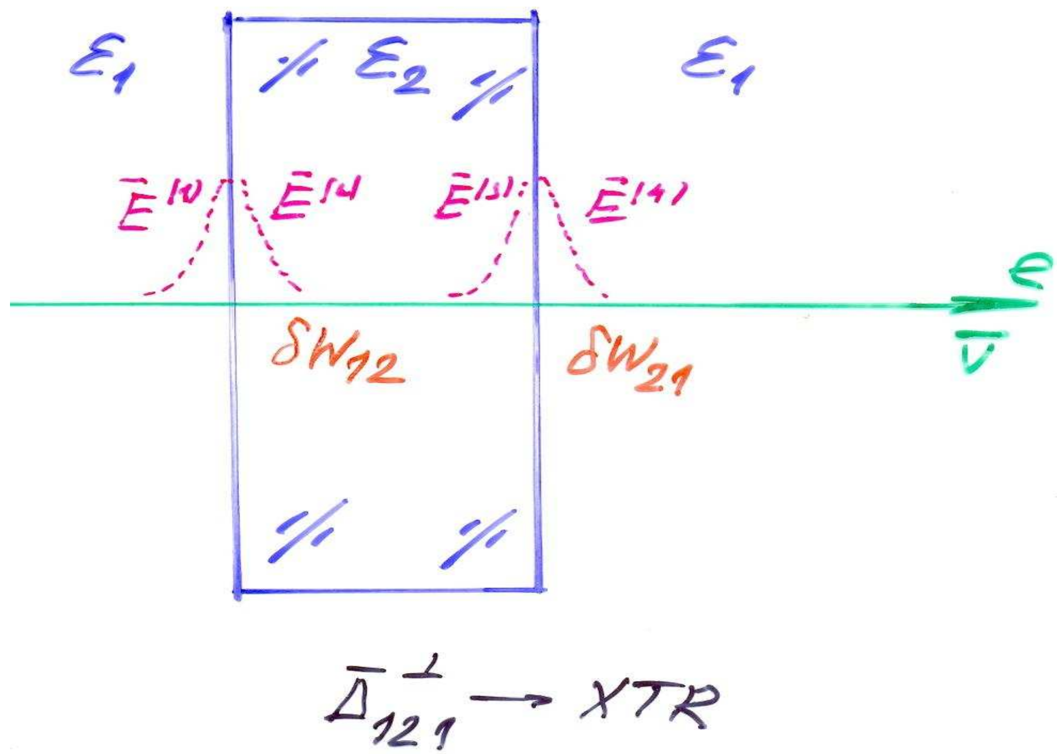

Figure 4: Diagram of a charged particle crossing the plate of medium 2 in the medium 1. Both media are considered to absorbing.

Let us consider limiting cases. If the second and the third media are identical, or $a = 0$, these relations are reduced to the expression describing the geometry with one interface. In the case that the first and the third media are identical, the expression for $R^{(123)}$ will be transformed to:

$$R_a^{(121)} = 2(Z_1 - Z_2)^2[1 - H_2(a)] = 2(Z_1 - Z_2)^2 \left[1 - \exp\left(-\frac{ia}{2Z_2}\right)\right].$$

The latter equation corresponds to a plate of the second medium with thickness a placed in the first medium. Note the presence of an additional factor of 2 which reflects the even number of interfaces.

In transparent media the latter relation becomes:

$$R_a^{(121)} = 4|L_1 - L_2|^2 \sin^2\left(\frac{a}{4L_2}\right),$$

in accordance with the standard theory.

In the limit of very thick plate we have

$$R^{(121)}_{a\to\infty} = 2(Z_1 - Z_2)^2.$$

It can be considered as XTR from two independent single interfaces between the first and the second media. Therefore the energy-angular distribution of XTR from a single interface reads:

$$\frac{d^2U^{(12)}}{\hbar d\omega\, d\theta^2} = \frac{d^2U^{(21)}}{\hbar d\omega\, d\theta^2} = \frac{\alpha}{\pi}\left(\frac{\omega}{c}\right)^2\theta^2 \mathrm{Re}\left\{(Z_1 - Z_2)^2\right\},$$

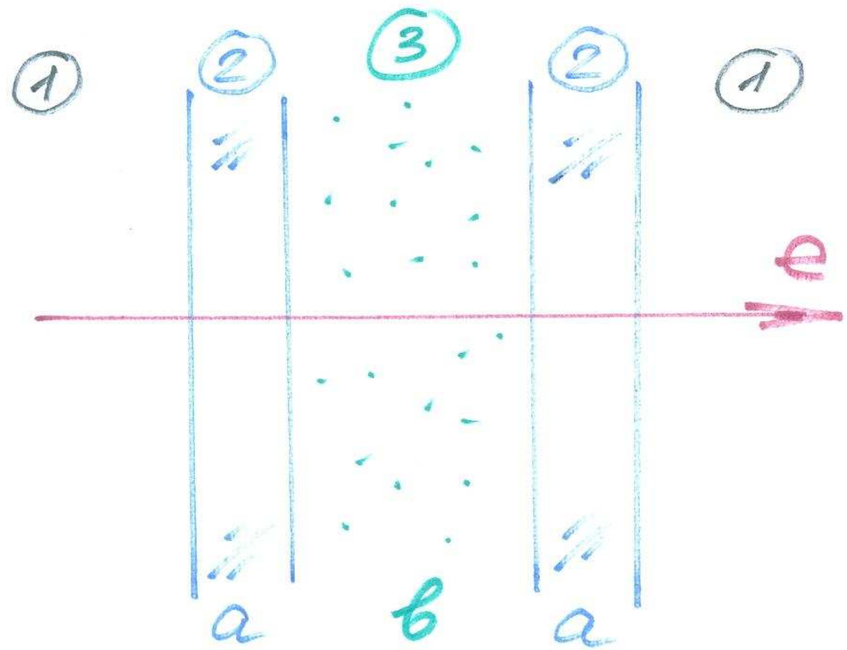

Figure 5: Diagram of a charged particle crossing a thin detector with media 1-2-3-2-1.

We now consider the geometry of a thin detector, when the charge serially crosses the interfaces 1-2 at $z = 0$, 2-3 at $z = a$, 3-2 at $z = a + b$, and 2-1 at $z = 2a + b$. Repeating the above considerations one gets:

$$\frac{d^2U^{(12321)}}{\hbar d\omega\, d\theta^2} = \frac{\alpha}{\pi}\left(\frac{\omega}{c}\right)^2 \theta^2 \mathrm{Re}\left\{R^{(12321)}\right\},$$

$$R^{(12321)} = 2\left\{(Z_1 - Z_2)^2[1 - H_2(2a)H_3(b)] + (Z_2 - Z_3)^2[1 - H_3(b)] + \right.$$
$$\left. + 2(Z_1 - Z_2)(Z_2 - Z_3)H_2(a)[1 - H_3(b)]\right\}.$$

In the case of $b = 0$ or $a = 0$ the latter equation will be reduced to the case of a plate of the second medium with thickness $2a$ or a plate of the third medium with thickness b, respectively, placed in the first medium:

$$R^{(12321)}_{b=0} = R^{121}_{2a} = 2(Z_1 - Z_2)^2[1 - H_2(2a)],$$

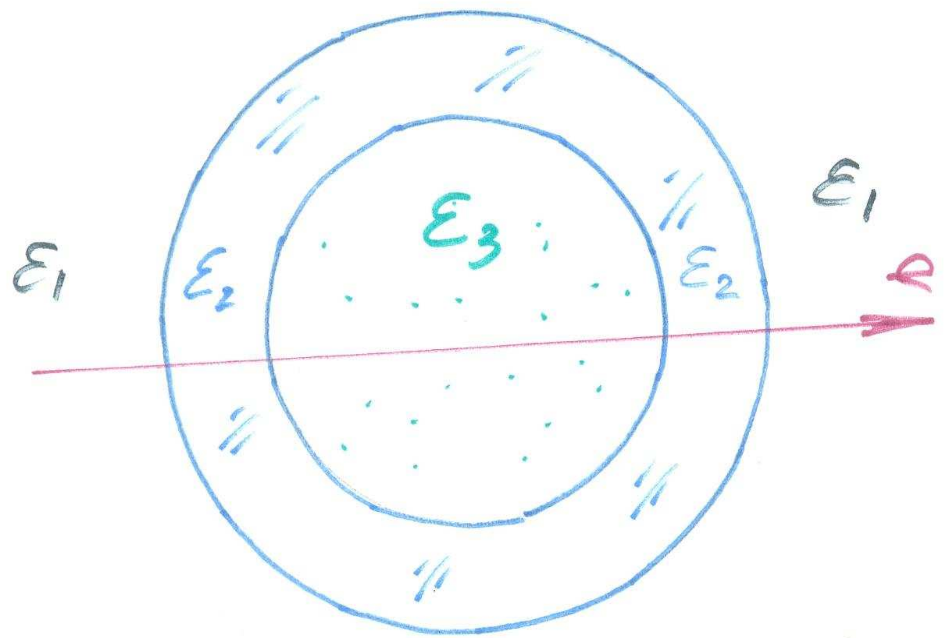

Figure 6: Diagram of a charged particle crossing the thin detector like straw tube. The particle crosses the interfaces; 1-2, 2-3, 3-2, 2-1.

$$R^{(12321)}_{a=0} = R^{131}_{b} = 2(Z_1 - Z_3)^2[1 - H_3(b)].$$

The same result will be, if the second and the third media are identical, namely, $R^{(12221)} = R^{121}_{2a+b}$. In the case of $b \to \infty$, this relation will describe the independent contribution from two detector windows:

$$R^{(12321)}_{b\to\infty} = 2R^{(123)} = 2[(Z_1 - Z_2)^2 + (Z_2 - Z_3)^2 + 2(Z_1 - Z_2)(Z_2 - Z_3)H_2(a)].$$

If the first and the third media are identical, $R^{(12321)}$-relation will be reduced to the case of two plates of the second medium with thickness a placed in the first medium with the gap equal to b:

$$R^{(12121)} = 2(Z_1 - Z_2)^2[2 - H_1(b) - 2H_2(a) + 2H_1(b)H_2(a) - H_1(b)H_2^2(a)].$$

Note that this relation can be applied to the description of metallic foils (Li,Be) when oxidation of surface layers is taken into account. It can also be applied to the description of dielectric foils covered with metallic layers.

4 The Most General Radiator

Let us consider a relativistic charge crossing $n-1$ interfaces between $n > 2$ different media located at arbitrary points $z = z_j$, $(j = 1, 2, \ldots, n)$. The electric fields will be the following:

$$\mathbf{E}^{(1)} \exp\left(-i\zeta_1 z\right) - \mathbf{B}^{(1)} \exp\left(i\frac{\omega}{v}z\right),$$

$$\mathbf{E}^{(2k-2)} \exp\left(i\zeta_k z\right) + \mathbf{E}^{(2k-1)} \exp\left(-i\zeta_k z\right) - \mathbf{B}^{(k)} \exp\left(i\frac{\omega}{v}z\right), \quad (k = 2, \ldots, n-1),$$

$$\mathbf{E}^{(2n-2)} \exp\left(i\zeta_n z\right) - \mathbf{B}^{(n)} \exp\left(i\frac{\omega}{v}z\right),$$

in the first, internal and last media, respectively. Note that, since the internal media $(k = 2, \ldots, n-1)$ are limited from both sides, the radiation fields can expand in both directions along the z-axis. The transverse conditions will be:

$$E_\tau^{(2k-1)} = \frac{E_n^{(2k-1)}}{\theta}, \quad E_\tau^{(2k)} = -\frac{E_n^{(2k)}}{\theta}, \quad k = 1, 2, \ldots, n-1.$$

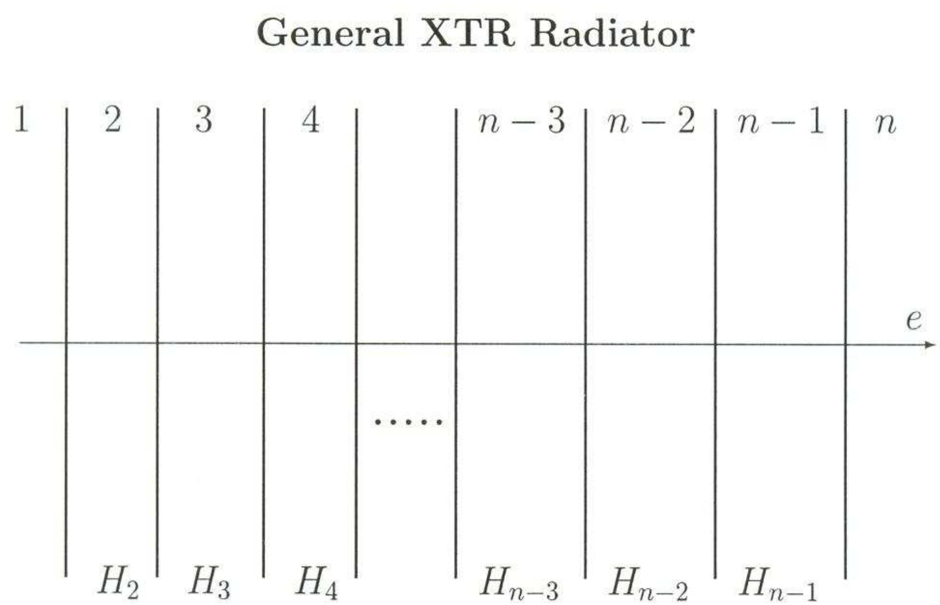

Figure 7: Diagram of a charged particle crossing the most general XTR radiator consisting of n media and $n-1$ interfaces between them. The medium thicknesses can fluctuate separately as described by the H_k ($k = 2, 3, \ldots, n-1$) values.

The boundary conditions at the $z = z_1$, read:

$$n: \; E_n^{(1)} \exp(-i\zeta_1 z_1) + A\theta^2 Z_1 \exp\left(i\frac{\omega}{v}z_1\right) =$$

$$= E_n^{(2)} \exp(i\zeta_2 z_1) + E_n^{(3)} \exp(-i\zeta_2 z_1) + A\theta^2 Z_2 \exp\left(i\frac{\omega}{v}z_1\right),$$

$$\tau: \; E_n^{(1)} \exp(-i\zeta_1 z_1) - A\theta^2 Z_1 \exp\left(i\frac{\omega}{v}z_1\right) =$$

$$= -E_n^{(2)} \exp(i\zeta_2 z_1) + E_n^{(3)} \exp(-i\zeta_2 z_1) - A\theta^2 Z_2 \exp\left(i\frac{\omega}{v}z_1\right).$$

From these equations we have again for $E_n^{(2)}$ (as for the plate case):

$$E_n^{(2)}(\mathbf{q}, \omega) = A\theta^2 (Z_1 - Z_2) H_2(-z_1), \quad (A = \frac{ie}{2\pi^2 c}),$$

where we introduce (as for the plate case) the convenient function $H_j(z)$:

$$H_j(z) = \exp\left[i\left(\zeta_j - \frac{\omega}{v}\right)z\right] \simeq \exp\left[-\frac{iz}{2Z_j}\right] =$$

$$= \exp\left(-\frac{z}{2l_j}\right)\left[\cos\left(\frac{z}{2L_j}\right) - i\sin\left(\frac{z}{2L_j}\right)\right],$$

with the following obvious properties:

$$H_j(z_1)H_j(z_2) = H_j(z_1+z_2), \quad \exp\left[i\left(\zeta_j - \zeta_k\right)z\right] = H_j(z)H_k(-z).$$

The boundary conditions at $z = z_{k-1}, \quad k = 3, \ldots, n-1$ read:

$$n: \; E_n^{(2k-3)}\exp(-i\zeta_{k-1}z_{k-1}) + E_n^{(2k-4)}\exp(i\zeta_{k-1}z_{k-1}) + A\theta^2 Z_{k-1}\exp\left(i\frac{\omega}{v}z_{k-1}\right) =$$

$$= E_n^{(2k-2)}\exp(i\zeta_k z_{k-1}) + E_n^{(2k-1)}\exp(-i\zeta_k z_{k-1}) + A\theta^2 Z_k\exp\left(i\frac{\omega}{v}z_{k-1}\right),$$

$$\tau: \; E_n^{(2k-3)}\exp(-i\zeta_{k-1}z_{k-1}) - E_n^{(2k-4)}\exp(i\zeta_{k-1}z_{k-1}) - A\theta^2 Z_{k-1}\exp\left(i\frac{\omega}{v}z_{k-1}\right) =$$

$$= -E_n^{(2k-2)}\exp(i\zeta_k z_{k-1}) + E_n^{(2k-1)}\exp(-i\zeta_k z_{k-1}) - A\theta^2 Z_k\exp\left(i\frac{\omega}{v}z_{k-1}\right).$$

The boundary conditions at $z = z_{n-1}$ read:

$$n: \; E_n^{(2n-3)} \exp(-i\zeta_{n-1}z_{n-1}) + E_n^{(2n-4)} \exp(i\zeta_{n-1}z_{n-1}) + A\theta^2 Z_{n-1} \exp\left(i\frac{\omega}{v}z_{n-1}\right) =$$

$$= E_n^{(2n-2)} \exp(i\zeta_n z_{n-1}) + A\theta^2 Z_n \exp\left(i\frac{\omega}{v}z_{n-1}\right),$$

$$\tau: \; E_n^{(2n-3)} \exp(-i\zeta_{n-1}z_{n-1}) - E_n^{(2n-4)} \exp(i\zeta_{n-1}z_{n-1}) - A\theta^2 Z_{n-1} \exp\left(i\frac{\omega}{v}z_{n-1}\right) =$$

$$= -E_n^{(2n-2)} \exp(i\zeta_n z_{n-1}) - A\theta^2 Z_n \exp\left(i\frac{\omega}{v}z_{n-1}\right).$$

The step-by-step subtraction of these relations results in
$E_n^{(2k-2)} \quad (k = 3, \ldots, n)$ being given by:

$$E_n^{(2k-2)}(\mathbf{q}, \omega) = A\theta^2 H_k(-z_{k-1}) \left[(Z_{k-1} - Z_k) + \sum_{i=1}^{k-2} (Z_i - Z_{i+1}) \prod_{j=i+1}^{k-1} H_j(a_j) \right],$$

where $a_k = z_k - z_{k-1}$, $(k = 2, \ldots, n-1)$ is the thickness of the k-th medium.

The Fourier image of the n-component of the radiation fields, expanding along the z-axis, reads:

$$E_n^{(1,\ldots,n)}(\mathbf{k},\omega) = \sum_{k=2}^{n-1} \int_{z_{k-1}}^{z_k} \frac{dz}{2\pi} E_n^{(2k-2)}(\mathbf{q},\omega) H_k(z) + \int_{z_{n-1}}^{\infty} \frac{dz}{2\pi} E_n^{(2n-2)}(\mathbf{q},\omega) H_n(z) =$$

$$= -\frac{e\theta^2}{2\pi^3 c}\left\{\sum_{k=2}^{n-1}\left[H_k(a_k)-1\right] Z_k \left[(Z_{k-1}-Z_k) + \sum_{i=1}^{k-2}(Z_i - Z_{i+1}) \prod_{j=i+1}^{k-1} H_j(a_j)\right] - \right.$$

$$\left. -Z_n\left[(Z_{n-1}-Z_n) + \sum_{i=1}^{n-2}(Z_i - Z_{i+1}) \prod_{j=i+1}^{n-1} H_j(a_j)\right]\right\}.$$

Taking half of the sum of the latter relation with $E_n^{(n,\ldots,1)}(\mathbf{k},\omega)$, with the substitution $k \leftrightarrows n-k+1, \quad (k = 1,\ldots,n)$, one gets, for the energy-angle distribution of the emitted XTR energy,

$$\frac{d^2U^{(1,\ldots,n)}}{\hbar d\omega\, d\theta^2} = \frac{\alpha}{\pi}\left(\frac{\omega}{c}\right)^2 \theta^2 \mathrm{Re}\left\{R^{(1,\ldots,n)}\right\},$$

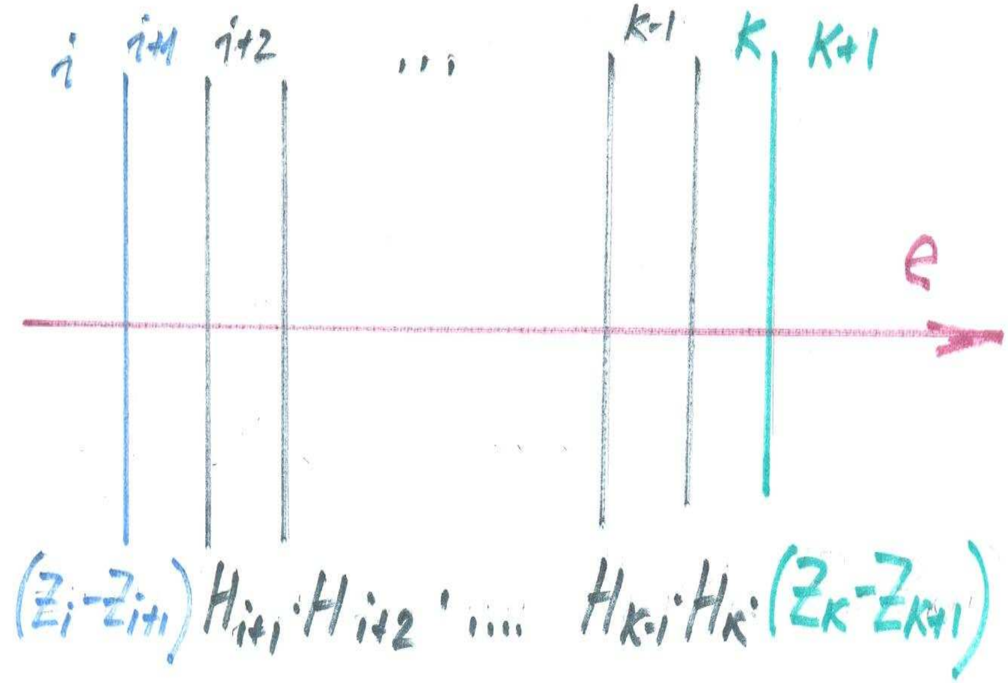

Figure 8: Contribution of correlation between $(i)-(i+1)$ and $(k)-(k+1)$ interfaces to $R^{(1,\dots,n)}$.

where the factor $R^{(1,\dots,n)}$ reflects the interference of contributions from different interfaces between media:

$$R^{(1,\dots,n)} = \sum_{i=1}^{n-1}(Z_i - Z_{i+1})^2 + 2\sum_{k=2}^{n-1}(Z_k - Z_{k+1})\sum_{i=1}^{k-1}(Z_i - Z_{i+1})\prod_{j=i+1}^{k} H_j(a_j).$$

These relations describe the energy-angle distribution of XTR photons emitted inside the *most general XTR radiator* consisting of arbitrary number $n > 2$ of different media with arbitrary thicknesses. One can see that the functions H_j have the physical meaning of correlation functions describing the interference between contributions from different interfaces.

Let us consider limiting cases. If $n = 3$ and the second and the third media are identical, or $a_2 = 0$, the general relations are reduced to the expression describing the geometry with one interface. In the case that the first and the third media are identical, the expression will be transformed to:

$$R^{(121)}_{a_2} = 2(Z_1 - Z_2)^2[1 - H_2(a_2)] = 2(Z_1 - Z_2)^2\left[1 - \exp\left(-\frac{ia_2}{2Z_2}\right)\right].$$

5 Models of Random Radiator Stack

In radiators based on foam or a fiber stack with inserted XTR detectors (straw tubes), XTR is measured from particles crossing different sets of the gap thicknesses. Since each gap contributes to the energy-angle distribution of the emitted XTR energy as not more than one H-factor, averaging over independently fluctuating gap thicknesses (for simplicity, we do not consider constrains due to finite radiator thickness) results in an expression for $\langle R^{(1,\dots,n)}\rangle$:

$$\langle R^{(1,\dots,n)}\rangle = \sum_{i=1}^{n-1}(Z_i - Z_{i+1})^2 + 2\sum_{k=2}^{n-1}(Z_k - Z_{k+1})\sum_{i=1}^{k-1}(Z_i - Z_{i+1})\prod_{j=i+1}^{k} H_j,$$

where H_j $(j = 2,\dots,n-1)$ are correlation functions averaged over the thickness distributions $p_j(z)$:

$$H_j = \langle H_j(z)\rangle = \int_0^\infty dz\, p_j(z)\exp\left[-\frac{iz}{2Z_j}\right], \quad \left(\int_0^\infty dz\, p_j(z) = 1\right).$$

Since the thicknesses of plates and gas gaps are positive and the H_j have exponential forms, it is convenient to represent the distribution $p_j(z)$ in the form of the Gamma distribution:

$$p_j(z) = \left(\frac{\nu_j}{\bar{z}_j}\right)^{\nu_j} \frac{z^{\nu_j - 1}}{\Gamma(\nu_j)} \exp\left[-\frac{\nu_j z}{\bar{z}_j}\right],$$

where Γ is the Euler gamma function, $\bar{z}_j$ is the mean thickness of the $j-$th medium in the radiator and $\nu_j > 0$ is a parameter which roughly describes the relative thickness fluctuations. In fact, the relative thickness fluctuation δ_j reads:

$$\delta_j = \frac{\sqrt{\langle (z_j - \bar{z}_j)^2 \rangle}}{\bar{z}_j} = \nu_j^{-1/2}, \quad (j = 1, 2).$$

Then one can derive analytically the values H_j. For example, in the general case of a gamma-distributed radiator:

$$H_j = \left[1 + i\frac{\bar{z}_j}{2Z_j\nu_j}\right]^{-\nu_j}.$$

This equation allows us to calculate the factor $R^{(1,\dots,n)}$ analytically and hence evaluate the energy-angle distribution of emitted XTR photons in a wide set of radiators with fluctuating gap thicknesses. For radiators with small irregularities, when $\nu_j > 100$, the direct calculation results in numerical instabilities and is time consuming. In this case the Gaussian distribution can be used for the averaging of the functions:

$$p_j(z) = \frac{1}{\sigma_j\sqrt{2\pi}}\exp\left[-\frac{(z-\bar{z}_j)^2}{2\sigma_j^2}\right],$$

where σ_j are the root-mean-squared fluctuations of the thickness in the j-th medium. Then for $\sigma_j \ll \bar{z}_j$:

$$H_j = \exp\left[-\frac{i\bar{z}_j}{2Z_j} - \frac{\sigma_j^2}{8Z_j^2}\right] = H_j(\bar{z}_j)\exp\left[-\frac{\sigma_j^2}{8Z_j^2}\right],$$

where $H_j(\bar{z}_j)$ corresponds to the case of the regular radiator ($p_j(z) = \delta(z-\bar{z}_j)$). In transparent media the second factor, $\exp(-\sigma_j^2/8Z_j^2)$, provides the numerical stability of the calculations.

6 Multiple Stacks of Plates

In practice, XTR is generated by special radiators consisting of a stack of foils (plastics like polypropylene or mylar) with gas gaps or a foam. Suppose we have n foils of the first medium (index 1) with thicknesses a_k, $k = 1, ..., n$ interspersed with gas gaps of the second medium (index 2) with thicknesses b_k, $k = 1, ..., n-1$. The relativistic charge serially intersects the sequence of gaps:

$$a_1, b_1, a_2, b_2, \ldots, b_{k-1}, a_k, b_k, \ldots, b_{n-1}, a_n.$$

Calculations similar to those considered in the previous sections result in the following expression for the energy-angle distribution of the emitted XTR energy:

$$\frac{d^2U^{(n)}}{\hbar d\omega\, d\theta^2} = \frac{\alpha}{\pi}\left(\frac{\omega}{c}\right)^2 \theta^2 \mathrm{Re}\left\{2(Z_1 - Z_2)^2 R^{(n)}\right\},$$

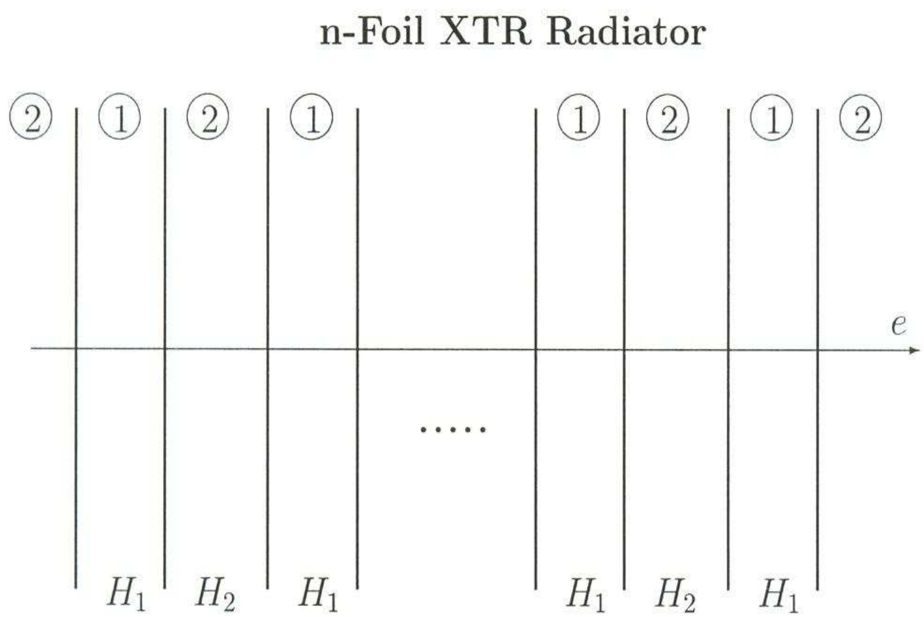

Figure 9: Diagram of a charged particle crossing a typical XTR radiator consisting of n foils of the first medium interspersed with gas gaps of the second medium. The foil and gas gap thicknesses can fluctuate separately as described by the H_1 and H_2 values.

where the stack factor $R^{(n)}$ can be represented as a sum of the contributions P_k from different plates in the stack:

$$R^{(n)} = \sum_{k=1}^{n} P_k, \quad \sum_{k=0}^{n} x^k = \frac{1-x^n}{1-x},$$

$$P_k = [1 - H_1(a_k)]\{1 - [1 - H_1(a_1)]H_2(b_1)H_1(a_2)\dots H_1(a_{k-1})H_2(b_{k-1}) -$$

$$-[1 - H_1(a_2)]H_2(b_2)H_1(a_3)\dots H_1(a_{k-1})H_2(b_{k-1}) -$$

$$- \dots - [1 - H_1(a_{k-1})]H_2(b_{k-1})\}.$$

These relations describe the spectrum of XTR emitted *inside* any general stack of n plates, where the thicknesses of the plates and gas gaps can be arbitrary (but fixed). They can be powerful for the determination of the optimal sequence of given XTR radiator materials for different experimental tasks of particle identification.

Let us consider the particular case when the thicknesses of all plates and gas gaps are equal to a and b, respectively. Calculating above expression for different n and using mathematical induction or performing direct algebra transformations one can obtain:

$$R^{(n)} = n\frac{[1-H_1(a)][1-H_2(b)]}{1-H_1(a)H_2(b)} + \frac{[1-H_1(a)]^2 H_2(b)\{1-[H_1(a)H_2(b)]^n\}}{[1-H_1(a)H_2(b)]^2}.$$

Note that in the case of transparent media this equation coincides with expression of standard theory, since the XTR energy loss inside a radiator will be equal to the XTR energy observed far from the radiator in the wave zone ($\bar{z}_1 = \bar{a}$ and $\bar{z}_2 = \bar{b}$). The mean number of XTR photons $\bar{N}_{in}$ emitted inside a radiator consisting two media with fluctuating thicknesses will be described by the following relation:

$$\frac{d^2\bar{N}_{in}}{\hbar d\omega\, d\theta^2} = \frac{2\alpha}{\pi\hbar c^2}\omega\theta^2 \mathrm{Re}\left\{\langle R^{(n)}\rangle\right\},$$

$$\langle R^{(n)}\rangle = (Z_1 - Z_2)^2\left\{n\frac{(1-H_1)(1-H_2)}{1-H} + \frac{(1-H_1)^2 H_2[1-H^n]}{(1-H)^2}\right\},$$

where $H = H_1 H_2$.

The integration of n-foil XTR radiator in respect to θ^2 can be simplified for the case of regular radiator ($\nu_{1,2} \to \infty$) with transparent in terms of XTR generation media, and $n \gg 1$ (G.M. Garibian, 1971). The frequency spectrum of emitted XTR photons reads:

$$\frac{d\bar{N}_{in}}{\hbar d\omega} = \int_0^{\sim 10\gamma^{-2}} d\theta^2 \frac{d^2\bar{N}_{in}}{\hbar d\omega\, d\theta^2} =$$

$$= \frac{4\alpha n}{\pi\hbar\omega}(C_1 + C_2)^2 \cdot \sum_{k=k_{min}}^{k_{max}} \frac{(k - C_{min})}{(k - C_1)^2 (k + C_2)^2} \sin^2\left[\frac{\pi t_1}{t_1 + t_2}(k + C_2)\right],$$

$$C_{1,2} = \frac{t_{1,2}(\omega_1^2 - \omega_2^2)}{4\pi c\omega},$$

$$C_{min} = \frac{1}{4\pi c}\left[\frac{\omega(t_1 + t_2)}{\gamma^2} + \frac{t_1\omega_1^2 + t_2\omega_2^2}{\omega}\right].$$

The sum $\sum_{k=k_{min}}^{k_{max}}$ is defined by terms with $k \gtrsim k_{min}$ corresponding to the region of $\theta \gtrsim 0$. Therefore k_{min} should be the nearest to C_{min} integer $k_{min} \geq C_{min}$. The value of k_{max} is defined by the maximum emitting angle $\theta_{max}^2 \sim 10\gamma^{-2}$. It can be evaluated as the integer part of:

$$C_{max} = C_{min} + \frac{\omega(t_1 + t_2)}{4\pi c}\frac{10}{\gamma^2}.$$

This value usually results in:

$$k_{max} - k_{min} \sim 10^2 \div 10^3 \gg 1,$$

however numerically, only few tens of terms contribute substantially to the sum, i.e. one can choose $k_{max} \sim k_{min} + 20$. Equation $d\bar{N}_{in}/\hbar d\omega$ corresponds to the spectrum of total number of photons emitted inside regular transparent radiator.

Therefore the mean interaction length, λ_{XTR}, of XTR process in this kind of radiators can be introduced as:

$$\lambda_{XTR} = n(t_1 + t_2) \left[\int_{\hbar\omega_{min}}^{\hbar\omega_{max}} \hbar d\omega \frac{d\bar{N}_{in}}{\hbar d\omega} \right]^{-1},$$

where $\hbar\omega_{min} \sim 1$ keV, and $\hbar\omega_{max} \sim 100$ keV for the majority of high energy physics experiments. The spectrum of total number of XTR photons after regular transparent radiator is defined by $d\bar{N}_{in}/\hbar d\omega$ with:

$$n \rightarrow n_{eff} = \frac{1 - \exp[-n(\sigma_1 t_1 + \sigma_2 t_2)]}{1 - \exp[-(\sigma_1 t_1 + \sigma_2 t_2)]},$$

where σ_1 and σ_2 are photo-absorption cross-sections corresponding to the photon frequency ω in the first and the second medium, respectively. With this correction taking into account the XTR photon absorption in the radiator, $d\bar{N}_{in}/\hbar d\omega$ corresponds to the results of C.W. Fabian and W. Struczinski, 1975 (see ALICE TDR).

7 Comparison with experimental data

These models were implemented in the framework of the GEANT4 simulation toolkit. In GEANT4 XTR generation *inside* radiators is described in the framework of the so-called parametrisation approach by a family of classes similar to that described in GEANT4 web site. The base abstract class G4VXTRdEdx is responsible for the creation of tables with integral energy and angular distributions of XTR photons. It has also the DoIt function providing XTR photon generation and moving the incident particle through a XTR radiator. Particular models like G4IrregularXTRdEdx realise the pure virtual function GetStackFactor, which calculates the response of XTR radiator $\langle R^{(n)} \rangle$.

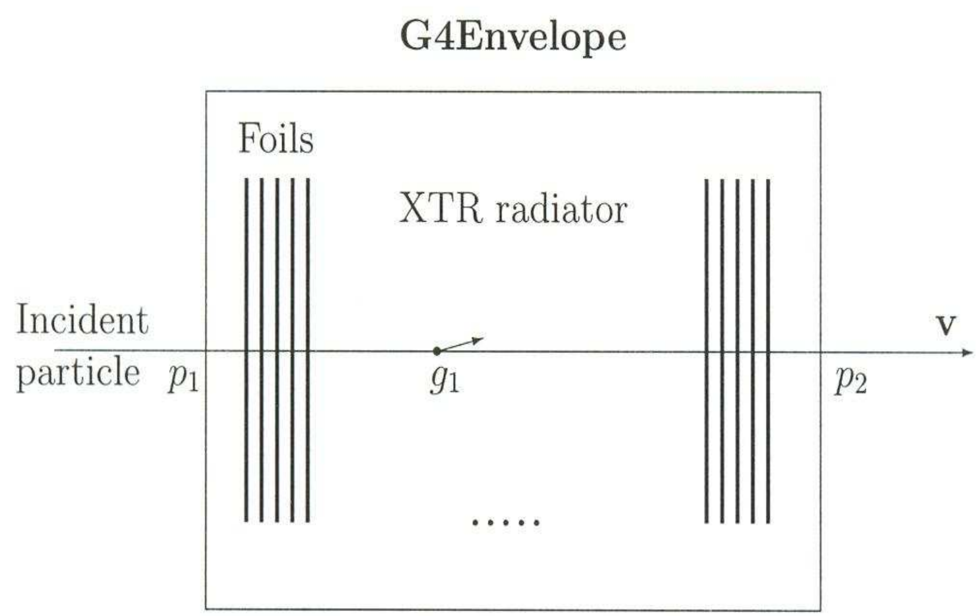

Figure 10: Working of the G4VXTRdEdx::DoIt function. An incident charged particle with Lorentz factor $\gamma \geqslant 100$ enters the logical volume G4Envelope at the point p_1 and exits at p_2. It moves along the direction given by the vector $\mathbf{v}$. Each XTR photon then will be considered as a secondary particle. The sum of the XTR photon energies is subtracted from the kinetic energy of the incident particle.

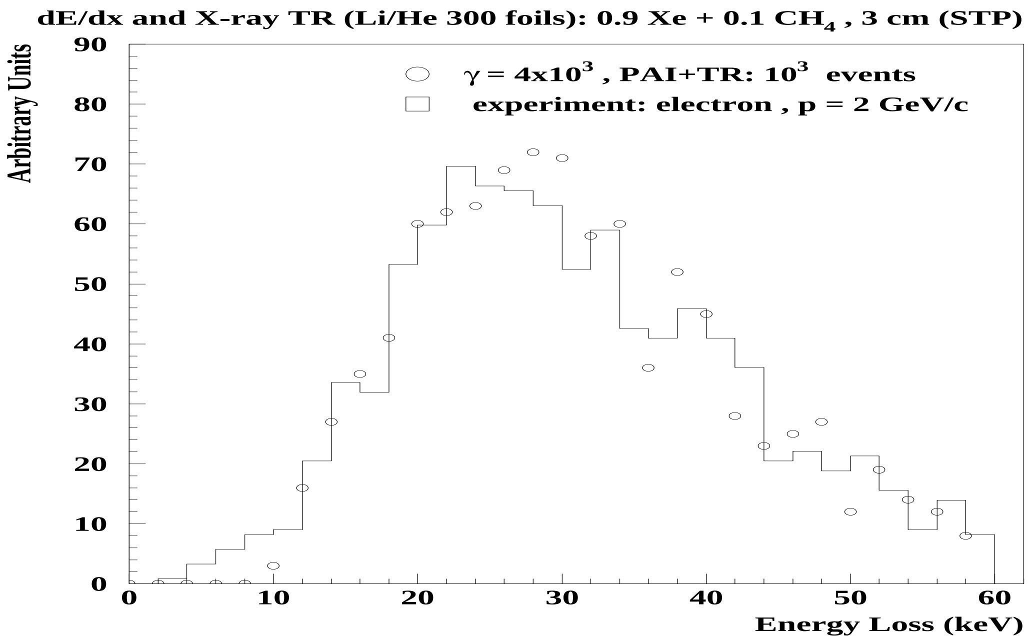

Figure 11: The ionisation energy loss distribution produced by electrons with a momentum of 3 GeV/c in a gas mixture $0.9\mathrm{Xe} + 0.1\mathrm{CH}_4$ with a thickness of 3 cm at pressure 1 atm. The histogram is the experimental data , open circles are simulation according to the PAI model and the XTR model (periodic 300 foils of 40 μm Li spaced 126 μm He).

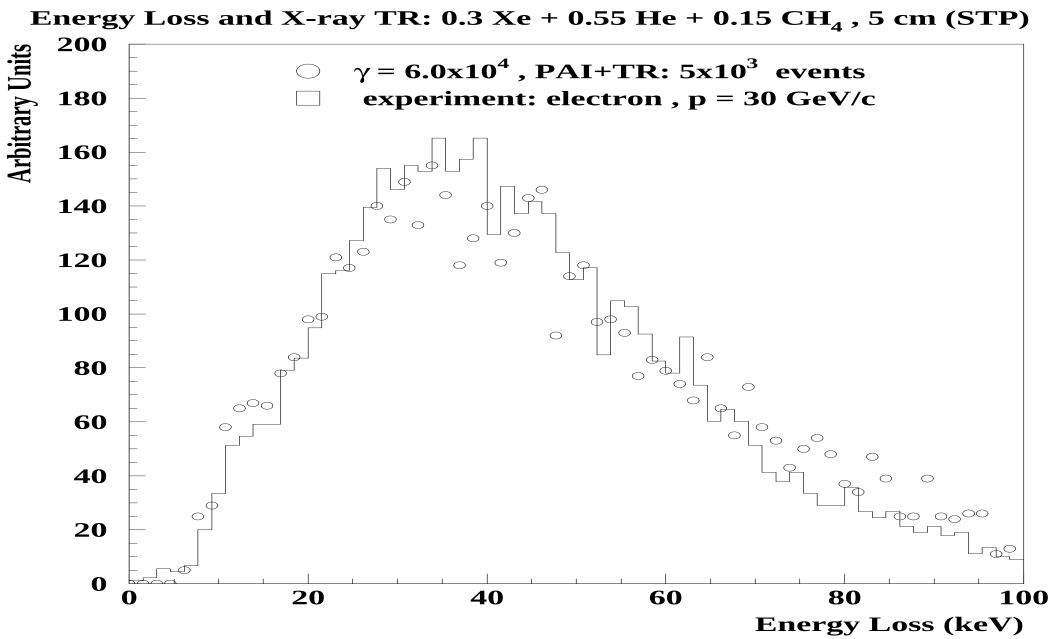

Figure 12: The ionisation energy loss distribution produced by electrons with a momentum of 30 GeV/c in a gas mixture 0.35Xe+0.55He+0.15CH$_4$ with a thickness of 5 cm at pressure 1 atm. The histogram is the experimental data , open circles are simulation according to the PAI model and the XTR model (periodic 350 foils 19 μm C_2H_2 spaced 600 μm CO_2).