Ionisation and its Fluctuations

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Abstract

Electrodynamics of relativistic charge in medium is considered in terms of its energy loss which is convenient for the simulation of emitted secondary particles (the medium excitations). The main relations are derived in details. Photo-absorption ionisation model is discussed with examples of calculations implemented in the framework of the GEANT4 toolkit. Energy loss fluctuations produced by relativistic charge crossing a medium layer with a fixed thickness are considered in terms of modified Poisson distribution. Ionisation fluctuations in very thin absorbers are derived.

1 Maxwell Equations in Medium

Electric $\mathbf{E}^e$ and magnetic $\mathbf{H}^e$ fields induced in vacuum by external electrical sources with the charge and current densities, ρ^e and $\mathbf{j}^e$, respectively, satisfy the Maxwell equations. In the Gauss system of units (c is the speed of light in vacuum, and $\hbar$ is the Planck constant) they are:

- $$\text{Gauss's law:} \quad \nabla \cdot \mathbf{E}^e = 4\pi\rho^e.$$
- $$\text{The absence of magnetic charge:} \quad \nabla \cdot \mathbf{H}^e = 0.$$
- $$\text{Faradey's law:} \quad \nabla \times \mathbf{E}^e + \frac{1}{c}\frac{\partial \mathbf{H}^e}{\partial t} = 0.$$
- $$\text{Ampere-Maxwell law:} \quad \nabla \times \mathbf{H}^e - \frac{1}{c}\frac{\partial \mathbf{E}^e}{\partial t} = \frac{4\pi}{c}\mathbf{j}^e.$$

In the analysis given below we'll confine our attention to the important special case of unordered, homogeneous, isotropic, and nongyrotropic equilibrium media, whose properties are invariant under translations and reflections of space and time, and also under rotations (in the rest frame of the medium as a whole). Perturbations of the medium by external charges are assumed small (linear electrodynamics). The medium is taken to be non-relativistic to the extent which this consistent with the existence of magnetism. The above restrictions are particularly convenient because they enable us to transform to the Fourier components of physical quantities ($i^2 = -1$):

$$F(\mathbf{r}, t) = \iint \frac{d\mathbf{k}\, d\omega}{(2\pi)^4} F(\mathbf{k}, \omega) \exp\left[i(\mathbf{kr} - \omega t)\right],$$

$$F(\mathbf{k}, \omega) = \iint d\mathbf{r}\, dt\, F(\mathbf{r}, t) \exp\left[-i(\mathbf{kr} - \omega t)\right].$$

Here ω and k are the wave frequency and the wave vector, respectively. For a real quantity $A(t, \mathbf{r})$: $A^*(\omega, \mathbf{k}) = A(-\omega, -\mathbf{k})$.

The Maxwell equations for Fourier components of fields in vacuum read:

$$\mathbf{k}\times\mathbf{H}^e + \frac{\omega}{c}\mathbf{E}^e = -\frac{4\pi i}{c}\mathbf{j}^e, \quad \mathbf{k}\cdot\mathbf{E}^e = -4\pi i\rho^e,$$

$$\mathbf{k}\times\mathbf{E}^e - \frac{\omega}{c}\mathbf{H}^e = 0, \quad \mathbf{k}\cdot\mathbf{H}^e = 0.$$

We now introduce transverse and longitudinal components (subscripts $\perp$ and $\|$, respectively) of vector quantity $\mathbf{V}$ relative to the wave vector $\mathbf{k}$ ($\mathbf{n}=\mathbf{k}/|\mathbf{k}|$):

$$\mathbf{V}_{\|} = \mathbf{k}\frac{\mathbf{k}\cdot\mathbf{V}}{k^2}, \quad \mathbf{V}_{\perp} = \mathbf{V}-\mathbf{V}_{\|}, \quad \mathbf{V} = \mathbf{n}(\mathbf{n}\cdot\mathbf{V}) + \mathbf{n}\times(\mathbf{n}\times\mathbf{V}).$$

One can see easily that:

$$\mathbf{k}\times\mathbf{V} = \mathbf{k}\times\mathbf{V}_{\perp}, \quad \mathbf{k}\cdot\mathbf{V} = \mathbf{k}\cdot\mathbf{V}_{\|}.$$

We'll use broadly (f.e., $\nabla\times(\nabla\times\mathbf{E})$, $|\mathbf{k}\times\mathbf{j}|^2$ ...):

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}),$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}).$$

The electromagnetic field in a medium is described by the electric field $\mathbf{E}$ and magnetic induction $\mathbf{B}$, i.e. by the average values of the microscopic electric and magnetic fields. The vectors $\mathbf{E}$ and $\mathbf{B}$ have a direct physical meaning: they appear in the Lorentz force:

$$\mathbf{F} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right),$$

acting on a classical test particle with the charge e moving with the velocity $\mathbf{v}$ in the medium. They also satisfy the Maxwell equations:

$$\mathbf{k} \times \mathbf{B} + \frac{\omega}{c}\mathbf{E} = -\frac{4\pi i}{c}\mathbf{j} = -\frac{4\pi i}{c}(\mathbf{j}^e + \mathbf{j}^i), \quad \mathbf{k} \cdot \mathbf{E} = -4\pi i \rho = -4\pi i(\rho^e + \rho^i),$$

$$\mathbf{k} \times \mathbf{E} - \frac{\omega}{c}\mathbf{B} = 0, \quad \mathbf{k} \cdot \mathbf{B} = 0,$$

where ρ and $\mathbf{j}$ are the total charge and current densities (satisfying the continuity equation, $\omega\rho = \mathbf{k} \cdot \mathbf{j}$). The indices e and i label external quantities and those induced in the medium by the external fields, respectively.

Instead of ρ^i, $\mathbf{j}^i$ it is common to introduce the electric displacement $\mathbf{D}$ and magnetic field $\mathbf{H}$ so that the equations with sources can be rewritten in the form:

$$\mathbf{k} \times \mathbf{H} + \frac{\omega}{c}\mathbf{D} = -\frac{4\pi i}{c}\mathbf{j}^e, \quad \mathbf{k} \cdot \mathbf{D} = -4\pi i \rho^e.$$

In contrast to $\mathbf{E}$ and $\mathbf{B}$, the quantities $\mathbf{D}$ and $\mathbf{H}$ do not have direct physical meaning (though $\mathbf{D}_{\|}$ and static $\mathbf{H}(\omega = 0)$ do have) because the transformation:

$$\mathbf{D} \to \mathbf{D} + \mathbf{k} \times \mathbf{N}, \quad \mathbf{H} \to \mathbf{H} - \frac{\omega}{c}\mathbf{N},$$

with arbitrary $\mathbf{N}$ does not alter the above equations. The Maxwell equations contain a number of redundant unknowns and must be complemented with constitutive (matter) relations that express individual properties of the medium. The latter usually relate ρ^i, $\mathbf{j}^i$ (or $\mathbf{D}$ and $\mathbf{H}$) and the fields $\mathbf{E}$ and $\mathbf{B}$.

The structure of these constitutive relations is determined by the symmetry properties of the medium. In linear electrodynamics:

$$\left(1-\frac{1}{\tilde{\mu}}\right)\mathbf{k}\times\mathbf{B}+(\tilde{\epsilon}-1)\,\frac{\omega}{c}\mathbf{E}_{\perp}=-\frac{4\pi i}{c}\mathbf{j}^{i}_{\perp},\quad (1-\epsilon)\,\mathbf{k}\cdot\mathbf{E}=-4\pi i\rho^{i}.$$

or

$$\mathbf{D}_{\|}=\epsilon\,\mathbf{E}_{\|},\quad \mathbf{D}_{\perp}=\tilde{\epsilon}\,\mathbf{E}_{\perp},\quad \mathbf{H}=\frac{1}{\tilde{\mu}}\mathbf{B}.$$

The quantities ϵ, $\tilde{\epsilon}$, and $\tilde{\mu}$ that parametrize these equations are, in general, integral linear operators acting in space and time (in the Fourier components, they are functions of ω and $\mathbf{k}$). Only two of them are independent. They correspond to the two types of field in the medium, i.e. the longitudinal field $\mathbf{E}_{\|}$ and one (because of strict relation of Faraday's law, $c\,\mathbf{k}\times\mathbf{E}_{\perp}=\omega\mathbf{B}$) transverse field $\mathbf{E}_{\perp}$ or $\mathbf{B}$. The quantities $\tilde{\epsilon}$ and $\tilde{\mu}$ have no independent meaning and can be varied arbitrarily but provided the following quantity η (normalized to unity in vacuum) remains constant.

To get η, we apply, $\mathbf{k}\times$ (Ampere-Maxwell law) in the medium and vacuum:

$$\frac{1}{\tilde{\mu}}\mathbf{k}\times(\mathbf{k}\times\mathbf{B})+\tilde{\epsilon}\,\frac{\omega}{c}\mathbf{k}\times\mathbf{E}_{\perp}=-\frac{4\pi i}{c}\mathbf{k}\times\mathbf{j}^{e}_{\perp}=\mathbf{k}\times(\mathbf{k}\times\mathbf{H}^{e})+\frac{\omega}{c}\mathbf{k}\times\mathbf{E}^{e}_{\perp}.$$

Using $c\,\mathbf{k}\times\mathbf{E}=\omega\mathbf{B}$ and $\mathbf{k}\cdot\mathbf{B}=0$, we have:

$$\left(-\frac{k^2}{\tilde{\mu}}+\tilde{\epsilon}\,\frac{\omega^2}{c^2}\right)\mathbf{B}=\frac{4\pi i}{c}\mathbf{k}\times\mathbf{j}^{e}_{\perp}=\left(-k^2+\frac{\omega^2}{c^2}\right)\mathbf{H}^{e}.$$

Therefore the value (see below, $\mathbf{H}^e=\eta\mathbf{B}$):

$$\eta=\left(\frac{k^2}{\tilde{\mu}}-\tilde{\epsilon}\,\frac{\omega^2}{c^2}\right)\left(k^2-\frac{\omega^2}{c^2}\right)^{-1}\simeq\text{constant},$$

since the response of both medium and vacuum is assumed to be linear $\sim\mathbf{j}^{e}_{\perp}$.
Accordingly there is a number of equivalent forms of constitutive equations.

The two most widely used correspond to the choice:

1.
$$\tilde{\epsilon} = \epsilon, \quad \tilde{\mu} = \mu,$$
so:
$$\mathbf{D} = \epsilon\,\mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}.$$
It can be used for radiation processes with ω up to optical frequencies.

2.
$$\tilde{\epsilon} = \epsilon_\perp = \epsilon + \left(1 - \frac{1}{\mu}\right)\frac{c^2k^2}{\omega^2}, \quad \tilde{\mu} = 1, \quad (\text{i.e.: } \mathbf{B} = \mathbf{H}).$$
so:
$$\mathbf{D}_\parallel = \epsilon\,\mathbf{E}_\parallel, \quad \mathbf{D}_\perp = \epsilon_\perp \mathbf{E}_\perp.$$
It is used for higher frequencies in plasma physics. Here we'll use it in ionisation models where $\hbar\omega > I_1$, where I_1 is the first ionisation potential.

Here ϵ is usual (longitudinal) permittivity, $\epsilon_\perp$ is transverse permittivity, and μ is magnetic permeability.

All such constitutive relations contain the single longitudinal parameter of the medium ϵ and differ by the form of the transverse parameter: μ or $\epsilon_\perp$. The most natural and convenient transverse parameter of the medium is, however, the quantity related to the μ or $\epsilon_\perp$ by:

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\eta = \left(\frac{k^2}{\mu} - \epsilon\,\frac{\omega^2}{c^2}\right) = \left(k^2 - \epsilon_\perp\,\frac{\omega^2}{c^2}\right),$$

$$\eta(\omega = 0) = \frac{1}{\mu}, \quad \eta(\omega \to \infty) = \epsilon_\perp.$$

In contrast to $\epsilon_\perp$ the quantity η does not have nonphysical singularity at $\omega = 0$, and, in contrast to μ, it does have a direct physical meaning at all frequencies. The parameters ϵ and η are associated with the particular form of constitutive equations, relating ρ^i, $\mathbf{j}^i$ not to the fields $\mathbf{E}$, $\mathbf{B}$, but to external sources:

$$3: \quad \rho^i = \left(\frac{1}{\epsilon} - 1\right)\rho^e, \quad \mathbf{j}^i_\perp = \left(\frac{1}{\eta} - 1\right)\mathbf{j}^e_\perp,$$

It is convenient to reduce these equations to (classical $\leftrightarrow$ quantum electrodynamics) a different form by introducing the potentials φ and $\mathbf{A}$ defined by: $\mathbf{E} = -i\mathbf{k}\varphi + i\omega\mathbf{A}/c$ and $\mathbf{B} = i\mathbf{k} \times \mathbf{A}$.

They ensure that, $c\mathbf{k} \times \mathbf{E} \equiv \omega \mathbf{B}$ and $\mathbf{k} \cdot \mathbf{B} \equiv 0$, become identities. If we adopt the gauge $\mathbf{k} \cdot \mathbf{A} = 0$ or $\mathbf{A} = \mathbf{A}_\perp$, we can replace Ampere-Maxwell and Gauss's laws with the following equations for the potentials:

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\mathbf{A}_\perp = \frac{4\pi}{c}\mathbf{j}_\perp, \quad k^2\varphi = 4\pi\rho.$$

The analogous equations:

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\mathbf{A}^e_\perp = \frac{4\pi}{c}\mathbf{j}^e_\perp, \quad k^2\varphi^e = 4\pi\rho^e.$$

determine the external potentials φ^e and $\mathbf{A}^e$ produced by the same external sources in vacuum. The constitutive relations can be expressed in terms of these potentials

$$\frac{4\pi}{c}\mathbf{j}_\perp = \left(k^2 - \frac{\omega^2}{c^2}\right)\frac{\mathbf{A}^e_\perp}{\eta(\omega, \mathbf{k})}, \quad 4\pi\rho = k^2\frac{\varphi^e}{\epsilon(\omega, \mathbf{k})}$$

which are distinguished by simplicity, lack of ambiguity, and clear physical meaning. In fact, it is readily seen that ϵ^{-1} and η^{-1} are the renormalisation factors reflecting the influence of medium in expression for longitudinal and transverse components of the photon Green function in the medium.

2 The Poynting's Theorem in Medium

We apply $\mathbf{E}\cdot$ operator to Ampere-Maxwell law and $\mathbf{H}\cdot$ operator to Faraday's laws following by their subtraction:

$$\mathbf{E}\cdot\left(\nabla\times\mathbf{H}=\frac{4\pi}{c}\mathbf{j}^e+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}\right)-\mathbf{H}\cdot\left(\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right).$$

$$\mathbf{E}\cdot\nabla\times\mathbf{H}-\mathbf{H}\cdot\nabla\times\mathbf{E}\equiv-\nabla\left(\mathbf{E}\times\mathbf{H}\right)=\frac{4\pi}{c}\mathbf{j}^e\cdot\mathbf{E}+\frac{1}{c}\left[\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right].$$

The above relation (right part) is called the Poynting's theorem in medium. It expresses the energy balance between external sources and electromagnetic fields in medium. Let us consider a point-like charge e moving in the medium along the trajectory $\mathbf{r}_o(t)$ with the velocity $\mathbf{v}_o(t)=\dot{\mathbf{r}}_o(t)$, where t is the time. The current density $\mathbf{j}^e$ reads: $\mathbf{j}^e(t)=e\mathbf{v}_o(t)\delta\left(\mathbf{r}-\mathbf{r}_o(t)\right)$. Integration of the Poynting's theorem in respect to volume surrounding the charge results in:

$$-e\mathbf{v}_o(t)\mathbf{E}(\mathbf{r}_o(t),t)=\frac{1}{4\pi}\int_V\left\{\mathbf{E}\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\frac{\partial\mathbf{B}}{\partial t}\right\}d\mathbf{r}+\oint_S\mathbf{S}d\mathbf{s},\quad\mathbf{S}=\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}.$$

3 Energy losses of arbitrarily moving charge

The calculation of energy loss is attractive in terms of simulation of secondary particles emitted from the primary trajectory of relativistic charge moving in a medium. These secondaries can be considered as transverse (photons, ...) and longitudinal (plasmons, ...) excitations of the medium. We hence expand the total energy loss $\bar{\Delta}$ to $\perp$ and $\|$ (relative to $\mathbf{k}$!) parts:

$$\bar{\Delta} = -\int_{-\infty}^{\infty} dt \int_{R_3} d\mathbf{r}\, \mathbf{j}(\mathbf{r},t) \cdot \left[\mathbf{E}_\perp(\mathbf{r},t) + \mathbf{E}_\|(\mathbf{r},t)\right] = \Delta_\perp + \Delta_\|,$$

where $\mathbf{j}(\mathbf{r},t)$ is the current density of external sources. The energy loss is convenient to represent as an integral in respect of the excitation energy $\hbar\omega > 0$ and its wave vector (or momentum $\hbar\mathbf{k}$). (It is in fact, implicit quantization!)

$$\bar{\Delta}_\perp = -\int_{-\infty}^{\infty} dt \int_{R_3} d\mathbf{r}\, \mathbf{j}(\mathbf{r},t) \cdot \mathbf{E}_\perp(\mathbf{r},t) = U_\perp = \frac{1}{4\pi}\int_{-\infty}^{\infty} dt \int_{R_3} d\mathbf{r}\, \mathbf{E}_\perp(\mathbf{r},t) \cdot \frac{\partial \mathbf{D}_\perp}{\partial t}.$$

$$\frac{1}{(2\pi)^8}\int d\omega\, d\mathbf{k} \int d\omega'\, d\mathbf{k}'\, \mathbf{j}(\mathbf{k}',\omega') \cdot \mathbf{E}_\perp(\mathbf{k},\omega) \exp\left[i(\mathbf{k}+\mathbf{k}')\mathbf{r} - i(\omega+\omega')t\right]$$

$$\bar{\Delta}_{\perp} = -\frac{2\pi^4}{(2\pi)^8}\int d\omega\, d\mathbf{k}\int d\omega^{'}\, d\mathbf{k}^{'}\, \mathbf{j}(\mathbf{k}^{'},\omega^{'})\cdot\mathbf{E}_{\perp}(\mathbf{k},\omega)\,\delta(\mathbf{k}+\mathbf{k}^{'})\delta(\omega+\omega^{'}),$$

$$\bar{\Delta}_{\perp} = -\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\omega\int_{K_3} d\mathbf{k}\,\mathbf{j}(-\mathbf{k},-\omega)\cdot\mathbf{E}_{\perp}(\mathbf{k},\omega), \quad \mathbf{j}(-\mathbf{k},-\omega) = \mathbf{j}^*(\mathbf{k},\omega),$$

$$\int_{-\infty}^{0} d\omega\,\mathbf{j}^*(\mathbf{k},\omega)\cdot\mathbf{E}_{\perp}(\mathbf{k},\omega) = (\omega\rightarrow-\omega) = \int_{0}^{\infty} d\omega\,\mathbf{j}^*(-\mathbf{k},-\omega)\cdot\mathbf{E}_{\perp}(-\mathbf{k},-\omega),$$

$$\bar{\Delta}_{\perp} = -\frac{1}{(2\pi)^4}\int_{0}^{\infty} d\omega\int_{K_3} d\mathbf{k}\,\left[\mathbf{j}^*(\mathbf{k},\omega)\cdot\mathbf{E}_{\perp}(\mathbf{k},\omega)+\mathbf{j}(\mathbf{k},\omega)\cdot\mathbf{E}^*_{\perp}(\mathbf{k},\omega)\right],$$

$$\bar{\Delta}_{\perp} = -\frac{2}{(2\pi)^4}\int_{0}^{\infty} d\omega\int_{K_3} d\mathbf{k}\,\mathrm{Re}\{\mathbf{j}^*(\mathbf{k},\omega)\cdot\mathbf{E}_{\perp}(\mathbf{k},\omega)\}.$$

We have obviously the same result for longitudinal energy loss:

$$\bar{\Delta}_{\parallel} = -\frac{2}{(2\pi)^4}\int_{0}^{\infty} d\omega\int_{K_3} d\mathbf{k}\,\mathrm{Re}\{\mathbf{j}^*(\mathbf{k},\omega)\cdot\mathbf{E}_{\parallel}(\mathbf{k},\omega)\}.$$

Our problem is reduced to the calculation of $\mathbf{E}_\perp(\mathbf{k},\omega)$ and $\mathbf{E}_\parallel(\mathbf{k},\omega)$ in terms of the charge and current densities of external sources $\mathbf{j}(\mathbf{k},\omega)$ and $\rho(\mathbf{k},\omega)$. We use for that the Maxwell equations for Fourier components:

$$\mathbf{k}\times\mathbf{H}+\epsilon\frac{\omega}{c}\mathbf{E}=-\frac{4\pi i}{c}\mathbf{j},\quad \epsilon\mathbf{k}\cdot\mathbf{E}=-4\pi i\rho,\quad \mathbf{k}\times\mathbf{E}=\frac{\omega}{c}\mu\mathbf{H}.$$

We have from the above equations:

$$\mathbf{E}=4\pi i\frac{\mu\frac{\omega}{c^2}\mathbf{j}-\frac{\rho}{\epsilon}\mathbf{k}}{k^2-\epsilon\mu\frac{\omega^2}{c^2}},\quad \mathbf{E}_\parallel=-4\pi i\frac{\rho}{\epsilon}\frac{\mathbf{k}}{k^2}.$$

Taking into account the continuity equation ($\mathbf{k}\cdot\mathbf{j}=\omega\rho$), we get:

$$\mathbf{E}_\perp=\mathbf{E}-\mathbf{E}_\parallel=4\pi i\mu\frac{\omega}{c^2}\frac{k^2\mathbf{j}-(\mathbf{k}\cdot\mathbf{j})\mathbf{k}}{k^2\left(k^2-\epsilon\mu\frac{\omega^2}{c^2}\right)}.$$

Therefore:

$$\bar{\Delta}_{\perp} = -\frac{2}{(2\pi)^4}\int_0^{\infty} d\omega \int_{K_3} d\mathbf{k}\,\mathrm{Re}\left\{4\pi i\mu\frac{\omega}{c^2}\frac{(\mathbf{k}\cdot\mathbf{k})(\mathbf{j}\cdot\mathbf{j}^*)-(\mathbf{k}\cdot\mathbf{j})(\mathbf{k}\cdot\mathbf{j}^*)}{k^2\left(k^2-\epsilon\mu\frac{\omega^2}{c^2}\right)}\right\}.$$

Taking into account ($\mathrm{Re}\{iz\} = -\mathrm{Im}\{z\}$) we have:

$$\bar{\Delta}_{\perp} = \frac{1}{2\pi^3}\int_0^{\infty}\omega\, d\omega\,\mathrm{Im}\left\{\int_{K_3} d\mathbf{k}\,\frac{\mu|\mathbf{k}\times\mathbf{j}(\mathbf{k},\omega)|^2}{c^2k^2\left(k^2-\epsilon\mu\frac{\omega^2}{c^2}\right)}\right\}.$$

Similarly (just substituting $\mathbf{E}_{\|}$ for $\mathbf{E}_{\perp}$), we have for $\Delta_{\|}$:

$$\bar{\Delta}_{\|} = \frac{1}{2\pi^3}\int_0^{\infty}\omega\, d\omega\,\mathrm{Im}\left\{\int_{K_3}\frac{d\mathbf{k}}{k^2}\,\frac{|\rho(\mathbf{k},\omega)|^2}{-\epsilon(\mathbf{k},\omega)}\right\}, \quad \omega|\rho(\mathbf{k},\omega)|^2 = \frac{|\mathbf{k}\cdot\mathbf{j}(\mathbf{k},\omega)|^2}{\omega}.$$

For the representation of $\bar{\Delta}_{\perp}$ in other transverse medium parameters, we use:

$$\frac{\mu}{\left(k^2-\epsilon\mu\frac{\omega^2}{c^2}\right)} = \frac{1}{\left(k^2-\epsilon_{\perp}\frac{\omega^2}{c^2}\right)} = \frac{1}{\left(k^2-\frac{\omega^2}{c^2}\right)\eta}.$$

Let a relativistic point-like charged particle with the charge e move along arbitrary trajectory $\mathbf{r}(t)$ with the velocity $\mathbf{v}(t)$ in an absorbing medium with the complex dielectric permittivity $\epsilon = \epsilon_1 + i\epsilon_2$ and magnetic permeability $\mu = \mu_1 + i\mu_2$. The corresponding Fourier component of the current density is:

$$\mathbf{j}(\mathbf{r}, t) = e\mathbf{v}(t)\delta\left[\mathbf{r} - \mathbf{r}(t)\right],$$

$$\mathbf{j}(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} dt \int_{R_3} d\mathbf{r}\, \mathbf{j}(\mathbf{r}, t) \exp\left[-i(\mathbf{kr} - \omega t)\right] = e\int_{-\infty}^{\infty} dt\, \mathbf{v}(t) \exp\left\{i\omega t - i\mathbf{kr}(t)\right\}.$$

We start to transform the total transverse energy loss $\Delta_\perp$, namely
$|\mathbf{k} \times \mathbf{j}(\mathbf{k}, \omega)|^2 = (k^2(\mathbf{j} \cdot \mathbf{j}^*) - \omega^2\rho\rho^*) =$

$$= e^2 \int dt_1 \int dt_2\, (\mathbf{k} \times \mathbf{v}(t_1)) \cdot (\mathbf{k} \times \mathbf{v}(t_2)) \exp\left\{i\omega(t_1 - t_2) - i\mathbf{k}[\mathbf{r}(t_1) - \mathbf{r}(t_2)]\right\},$$

substituting, $t = t_2$, $\tau = t_1 - t_2$, we get:

$$= e^2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tau\, (\mathbf{k} \times \mathbf{v}(t + \tau)) \cdot (\mathbf{k} \times \mathbf{v}(t)) \exp\left\{i\omega\tau - i\mathbf{k}[\mathbf{r}(t + \tau) - \mathbf{r}(t)]\right\}.$$

We can consider now the variable t as the time along the charge trajectory

The mean number of photons $\bar{N}_\perp$ emitted into unit solid angle Ω, per unit energy $\hbar\omega$, in unit time t reads (at a given frequency: $d\bar{\Delta}_\perp \sim \hbar\omega\, d\bar{N}_\perp$):

$$\frac{d^3\bar{N}_\perp(t)}{\hbar\, d\omega\, dt\, d\Omega} = \frac{1}{\hbar\omega}\frac{d^3\bar{\Delta}_\perp(t)}{\hbar\, d\omega\, dt\, d\Omega} = \frac{\alpha}{2\pi^3\hbar c}\mathrm{Im}\left\{\int_0^\infty \frac{\mu(\omega)\, dk}{\left[k^2 - \epsilon(\omega)\mu(\omega)\dfrac{\omega^2}{c^2}\right]}\right.$$

$$\left.\int_{-\infty}^{\infty} d\tau\ \left[k^2\mathbf{v}(t+\tau)\mathbf{v}(t) - \omega^2\right]\exp\left\{i\omega\tau - i\mathbf{k}[\mathbf{r}(t+\tau) - \mathbf{r}(t)]\right\}\right\},$$

where $\alpha = e^2/\hbar c$ is the fine structure constant, Ω is the solid angle defining the direction of $\mathbf{k}$ versus $\mathbf{v}(t)$: $d\mathbf{k} = k^2 dk\, d\Omega$. It is, in fact, the double differential cross-section for emission of transverse medium excitations from an arbitrary charged particle trajectory, $\mathbf{r}(t)$.

- No radiation recoil effects on the charge trajectory, $\mathbf{r}(t)$.
- For the particle charge $q = ze$, the result will $\sim z^2\alpha$...
- The condition, $\omega > 0$, results in Im(...).
- Relation can be integrated in respect of Ω for any $\mathbf{r}(t)$.

We use known integral in respect to solid angle, $\mathbf{k} = k\mathbf{n}$:

$$\int_{4\pi} d\Omega \exp\{-i\mathbf{n}k[\mathbf{r}(t+\tau) - \mathbf{r}(t)]\} = 4\pi \frac{\sin[k|\mathbf{r}(t+\tau) - \mathbf{r}(t)|]}{k|\mathbf{r}(t+\tau) - \mathbf{r}(t)|}.$$

Then the spectral intensity or differential cross-section read:

$$\frac{d^2\bar{N}_\perp(t)}{\hbar\, d\omega\, dt} = \frac{1}{\hbar\omega}\frac{d^2\bar{\Delta}_\perp(t)}{\hbar\, d\omega\, dt} = \frac{2\alpha}{\pi^2\hbar c}\mathrm{Im}\left\{\int_0^\infty \frac{\mu(\omega)\, dk}{k\left[k^2 - \epsilon(\omega)\mu(\omega)\frac{\omega^2}{c^2}\right]}\right.$$

$$\left.\int_{-\infty}^{\infty} d\tau\, \frac{\left[k^2\mathbf{v}(t+\tau)\mathbf{v}(t) - \omega^2\right]}{|\mathbf{r}(t+\tau) - \mathbf{r}(t)|}\exp[i\omega\tau]\sin[k|\mathbf{r}(t+\tau) - \mathbf{r}(t)|]\right\}.$$

These general relations allow us to consider many particular practical cases, namely Cherenkov and Doppler radiations, synchrotron radiation, prompt bremsstrahlung, the influence of multiple scattering on the emission of optical photons etc. We fix the trajectory $\mathbf{r}(t)$ mode only.

4 Photo Absorption Ionisation model

We consider in this section the simplified derivation of Photo Absorption Ionisation (PAI) model in terms of classical electrodynamics (Allison and Cobb, 1980). We start from the general relations for transverse and longitudinal excitations produced by relativistic charge e moving in the medium with constant velocity v. The mean number of photons $\bar{N}_\perp$ emitted into unit solid angle Ω, per unit energy $\hbar\omega$, in unit time t reads (at a given frequency: $d\bar{\Delta}_\perp \sim \hbar\omega\, d\bar{N}_\perp$):

$$\frac{d^3\bar{N}_\perp(t)}{\hbar\, d\omega\, dt\, d\Omega} = \frac{\alpha}{2\pi^3\hbar c}\mathrm{Im}\left\{\int_0^\infty \frac{dk}{\left[k^2 - \epsilon_\perp(\mathbf{k},\omega)\frac{\omega^2}{c^2}\right]}\right.$$

$$\left.\int_{-\infty}^{\infty} d\tau\, \left[k^2\mathbf{v}(t+\tau)\mathbf{v}(t) - \omega^2\right]\exp\left\{i\omega\tau - i\mathbf{k}[\mathbf{r}(t+\tau) - \mathbf{r}(t)]\right\}\right\},$$

where $\alpha = e^2/\hbar c$ is the fine structure constant, Ω is the solid angle defining the direction of $\mathbf{k}$ versus $\mathbf{v}(t)$: $d\mathbf{k} = k^2 dk\, d\Omega$.

For the case of movement with the constant velocity $\mathbf{v}$ ($\mathbf{r}(t) = \mathbf{v}t$) the integral with respect to τ can be calculated easily:

$$\int_{-\infty}^{\infty} d\tau \, \left[k^2 v^2 - \omega^2\right] \exp\{i(\omega - \mathbf{k}\cdot\mathbf{v})\tau\} = 2\pi k^2 v^2 \sin^2\theta \delta(\mathbf{kv} - \omega),$$

where θ is the angle between $\mathbf{v}$and $\mathbf{k}$. The delta function allows us to calculate the integral in respect to θ (rather then k!). Taking into account $d\Omega = 2\pi d\cos\theta$ (no φ-dependence of integrated function) we have:

$$\delta(...) \to \frac{1}{kv}, \quad \cos\theta = \frac{\omega}{kv} \geqslant 0, \quad k = \frac{\omega}{\omega\cos\theta} \geqslant \frac{\omega}{v},$$

$$\frac{d^2\bar{N}_\perp}{\hbar\, d\omega\, dt} = \frac{2\alpha}{\pi\hbar c}\mathrm{Im}\left\{\int_{\omega/v}^{\infty} \frac{kv\left(1 - \frac{\omega^2}{k^2v^2}\right)dk}{\left[k^2 - \epsilon_\perp(k,\omega)\frac{\omega^2}{c^2}\right]}\right\}, \quad dx = vdt, \quad k \to k^2,$$

$$\frac{d^2\bar{N}_\perp}{\hbar\, d\omega\, dx} = \frac{\alpha}{\pi\hbar c}\mathrm{Im}\left\{\int_{(\omega/v)^2}^{\infty} \frac{\left(1 - \frac{\omega^2}{k^2v^2}\right)dk^2}{\left[k^2 - \epsilon_\perp(k,\omega)\frac{\omega^2}{c^2}\right]}\right\}, \quad t = \frac{k^2v^2}{\omega^2},$$

According to the PAI model approach, the transverse dielectric permittivity is supposed to satisfy the dipole approximation in all resonance region of atomic frequencies, i.e to depend for all k on ω only:

$$\epsilon_{\perp 2}(k,\omega) \equiv \epsilon_{\perp 2}(\omega) \simeq \frac{Nc}{\omega}\sigma_\gamma(\omega), \quad \epsilon_{\perp 1}(k,\omega) \simeq \epsilon_{\perp 1}(\omega) = P\int_0^\infty \frac{\omega^{'}\epsilon_{\perp 2}(\omega^{'})\,d\omega^{'}}{\omega^{'2}-\omega^2},$$

where the integral in respect of $\omega^{'}$ is treated as the principal value.

$$\mathrm{Im}\left\{\int_1^\infty \frac{\left(1-\frac{1}{t}\right)dt}{[t-\epsilon_\perp(\omega)\beta^2]}\right\}, \quad \epsilon_\perp(\omega)\beta^2 = a,$$

$$\mathrm{Im}\left\{\int_1^\infty\left[\left(1-\frac{1}{a}\right)\frac{1}{t-a}+\frac{1}{at}\right]dt\right\} = \mathrm{Im}\left\{\left[\left(1-\frac{1}{a}\right)\ln(T-a)+\frac{1}{a}\ln T\right]\right\}_{T\to\infty} -$$

$$-\mathrm{Im}\left\{\left[\left(1-\frac{1}{a}\right)\ln(1-a)+\frac{1}{a}\ln 1\right]\right\} = \mathrm{Im}\left\{\left(1-\frac{1}{\epsilon_\perp(\omega)\beta^2}\right)\ln\frac{1}{1-\epsilon_\perp(\omega)\beta^2}\right\}.$$

Therefore the transverse spectrum finally reads:

$$\frac{d^2\bar{N}_\perp(t)}{\hbar\, d\omega\, dx} = \frac{\alpha}{\hbar c}\frac{1}{\pi}\mathrm{Im}\left\{\left(1-\frac{1}{\epsilon_\perp(\omega)\beta^2}\right)\ln\frac{1}{1-\epsilon_\perp(\omega)\beta^2}\right\}.$$

It is exactly the number of Cherenkov photons emitted from unit trajectory length into unit energy. We actually derived:

$$\frac{d^2\bar{N}_\perp(t)}{\hbar\, d\omega\, dx} = \frac{2\alpha}{\hbar c}\frac{1}{\pi}\mathrm{Im}\left\{\int_0^1 \frac{\sin^2\theta\, d\cos\theta}{\cos\theta}\frac{1}{1-\epsilon_\perp(\omega)\beta^2\cos^2\theta}\right\}, \quad t=\cos^2\theta,$$

$$\mathrm{Im}\left\{\int_0^1 \frac{(1-t)\,dt}{t(1-at)}\right\} = \mathrm{Im}\left\{\int_0^1 \left[\frac{1}{t}+\frac{1-a}{at-1}\right]dt\right\} = \mathrm{Im}\left\{\int_0^1 \frac{1-a}{at-1}\,dt\right\} =$$

$$= \mathrm{Im}\left\{\int_0^1 \frac{\frac{1}{a}-1}{t-\frac{1}{a}}\,dt\right\} = \mathrm{Im}\left\{\left(\frac{1}{a}-1\right)\left[\ln\left(1-\frac{1}{a}\right)-\ln\left(-\frac{1}{a}\right)\right]\right\} =$$

$$= \mathrm{Im}\left\{\left(1-\frac{1}{\epsilon_\perp(\omega)\beta^2}\right)\ln\frac{1}{1-\epsilon_\perp(\omega)\beta^2}\right\}.$$

The mean number of longitudinal excitations (mainly electrons for atomic frequencies) $\bar{N}_{\|}$ emitted into unit solid angle Ω, per unit energy $\hbar\omega$, in unit time t reads (at a given frequency: $d\bar{\Delta}_{\|} \sim \hbar\omega\, d\bar{N}_{\|}$):

$$\frac{d^3\bar{N}_{\|}(t)}{\hbar\, d\omega\, dt\, d\Omega} = \frac{\alpha}{2\pi^3\hbar c}\mathrm{Im}\left\{\int_0^\infty \frac{dk}{-\epsilon(\mathbf{k},\omega)\dfrac{\omega^2}{c^2}}\right.$$

$$\left.\int_{-\infty}^{\infty} d\tau\ [\mathbf{k}\cdot\mathbf{v}(t+\tau)]\,[\mathbf{k}\cdot\mathbf{v}(t)]\exp\left\{i\omega\tau - i\mathbf{k}[\mathbf{r}(t+\tau)-\mathbf{r}(t)]\right\}\right\}.$$

For the case of movement with the constant velocity $\mathbf{v}$ ($\mathbf{r}(t) = \mathbf{v}t$) the integral with respect to τ is calculated similarly:

$$\int_{-\infty}^{\infty} d\tau\, k^2 v^2 \cos^2\theta\, \exp\{i(\omega - \mathbf{k}\cdot\mathbf{v})\tau\} = 2\pi k^2 v^2 \cos^2\theta\, \delta(\mathbf{kv}-\omega),$$

$$\delta(...) \to \frac{1}{kv}, \quad \cos\theta = \frac{\omega}{kv} \geqslant 0, \quad k = \frac{\omega}{\omega\cos\theta} \geqslant \frac{\omega}{v},$$

Therefore after integration in respect of θ we have:

$$\frac{d^2\bar{N}_{\|}}{\hbar\, d\omega\, dx} = \frac{\alpha}{\hbar c}\frac{2}{\pi}\mathrm{Im}\left\{\int_{\omega/v}^{\infty}\frac{dk}{k}\frac{1}{-\epsilon(k,\omega)\beta^2}\right\},$$

$$\frac{1}{-\epsilon} = \frac{-\epsilon_1 + i\epsilon_2}{|\epsilon|^2} \simeq i\epsilon_2, \quad |\epsilon|^2 \simeq 1, \quad \mathrm{Im}\{i\epsilon_2\} = \epsilon_2,$$

The photo-absorption model approximation reads for $\epsilon(k,\omega)$:

$$\epsilon_2(\omega,k) \equiv \frac{Nc}{\omega}\left[\sigma_\gamma(\omega)H\left(\omega - \frac{\hbar k^2}{2m}\right) + \delta\left(\omega - \frac{\hbar k^2}{2m}\right)\int_0^{\omega}\sigma_\gamma(\omega')\,d\omega'\right],$$

where H is the Heaviside unit step function. The number of longitudinal excitations now reads (we introduce new variable $t = k^2$):

$$\frac{d^2\bar{N}_{\|}}{\hbar\, d\omega\, dx} = \frac{\alpha}{\hbar}\frac{N}{\pi\beta^2}\int\limits_{(\omega/v)^2}^{\infty}\frac{dt}{t}\left[\frac{\sigma_\gamma(\omega)}{\omega}H\left(\omega - \frac{\hbar t}{2m}\right) + \frac{1}{\omega}\delta\left(\omega - \frac{\hbar t}{2m}\right)\int\limits_0^{\omega}\sigma_\gamma(\omega')\,d\omega'\right].$$

We note that:

$$\frac{\omega^2}{v^2} < t < \frac{2m\omega}{\hbar}, \quad \text{since,} \quad \hbar\omega \lesssim 2mv^2 \sim 2mc^2 \sim 1\,\text{MeV},$$

then the first term is proportional to:

$$\int_{(\omega/v)^2}^{\infty} \frac{dt}{t} H\left(\omega - \frac{\hbar t}{2m}\right) = \int_{(\omega/v)^2}^{2m\omega/\hbar} \frac{dt}{t} = \ln\frac{2mv^2}{\hbar\omega},$$

and the second one is proportional to:

$$\delta(...) \rightarrow \frac{2m}{\hbar}, \quad t = \frac{2m\omega}{\hbar},$$

$$\int_{(\omega/v)^2}^{\infty} \frac{dt}{t}\delta\left(\omega - \frac{\hbar t}{2m}\right) = \frac{2m}{\hbar}\frac{\hbar}{2m\omega} = \frac{1}{\omega}.$$

Therefore we have finally for the spectrum of longitudinal excitations (δ-electrons):

$$\frac{d^2\bar{N}_{||}}{\hbar d\omega\, dx} = \frac{\alpha N}{\pi\hbar\beta^2}\left[\frac{\sigma_\gamma(\omega)}{\omega}\ln\frac{2mv^2}{\hbar\omega} + \frac{1}{\omega^2}\int_0^{\omega}\sigma_\gamma(\omega')\,d\omega'\right],$$

Combining these results we have for the total number of ionizing collisions produced by relativistic charge per unit trajectory length in unit energy interval (near $\hbar\omega$) according to the PAI model:

$$\frac{d^2\bar{N}}{\hbar d\omega\, dx} = \frac{\alpha}{\pi\hbar c}\left\{\mathrm{Im}\left[\left(1 - \frac{1}{\epsilon_\perp(\omega)\beta^2}\right)\ln\frac{1}{1-\epsilon_\perp(\omega)\beta^2}\right] + \right.$$

$$\left. + \frac{Nc}{\beta^2}\left[\frac{\sigma_\gamma(\omega)}{\omega}\ln\frac{2mv^2}{\hbar\omega} + \frac{1}{\omega^2}\int_0^\omega \sigma_\gamma(\omega^{'})\, d\omega^{'}\right]\right\},$$

$$\epsilon_{\perp 2}(\omega) \simeq \frac{Nc}{\omega}\sigma_\gamma(\omega), \quad \epsilon_{\perp 1}(k,\omega) \simeq \epsilon_{\perp 1}(\omega) = P\int_0^\infty \frac{\omega^{'}\epsilon_{\perp 2}(\omega^{'})\, d\omega^{'}}{\omega^{'2} - \omega^2}.$$

We see that the ionisation spectrum is expressed in terms of photo-absorption cross-section (PAI) which is well and reliably tabulated for the majority of chemical elements. The first term express the transverse cross-section (Cherenkov photons). The second (comes from resonance region where internal electron motion is much smaller compared to its binding energy) and third (represents Rutherford scattering from those electrons than are quasi-free for an energy transfer $\hbar\omega$) are known as the longitudinal cross-section.

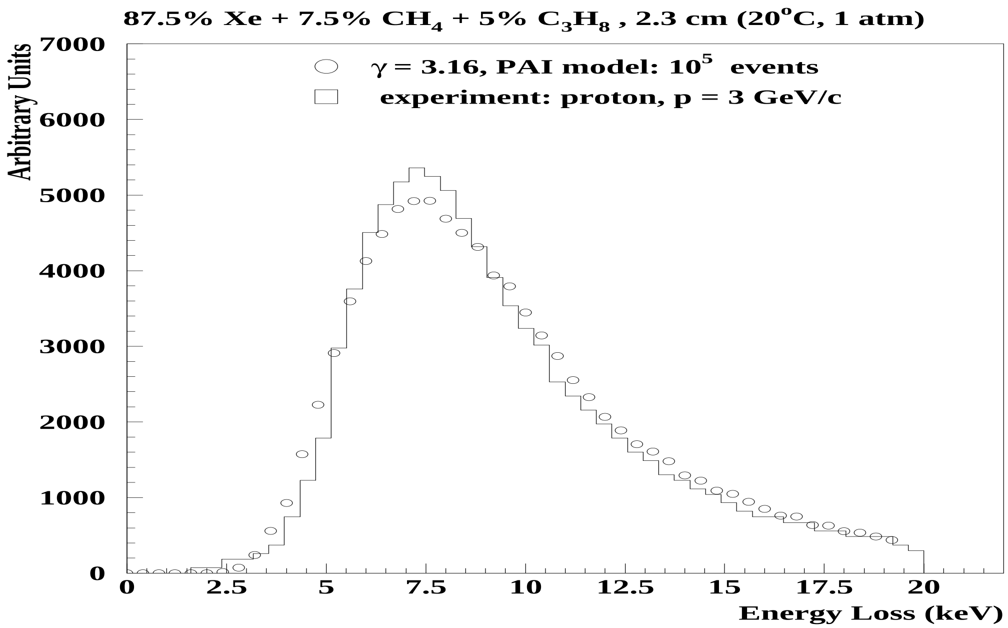

Figure 1: The ionisation energy loss distribution produced by protons with the momentum of 3 GeV/c in the gas mixture $87.5\% Xe + 7.5\% CH_4 + 5\% C_3H_8$ with the thickness of 2.3 cm (20 ^{o}C, 1 atm). Histogram is the experimental data , open circles are simulation according to the PAI model.

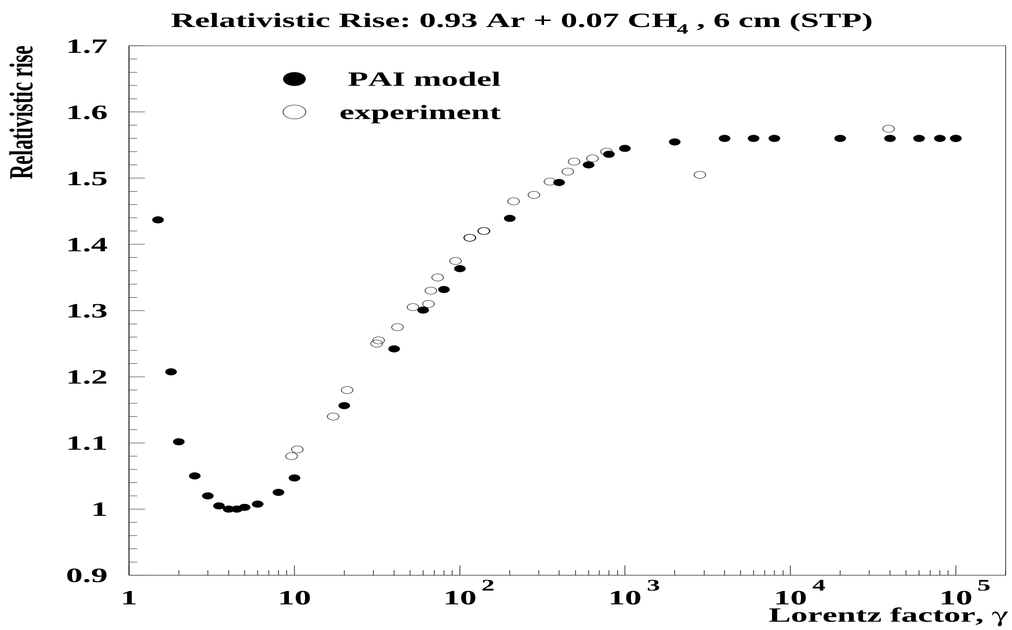

Figure 2: Relativistic rise of the most probable ionisation energy loss in the gas mixture$93\% Ar + 7\% CH_4$ with the thickness of 6 cm (STP). Open circles are the experimental data , closed circles are simulation according to the PAI model. It has natural density effect.

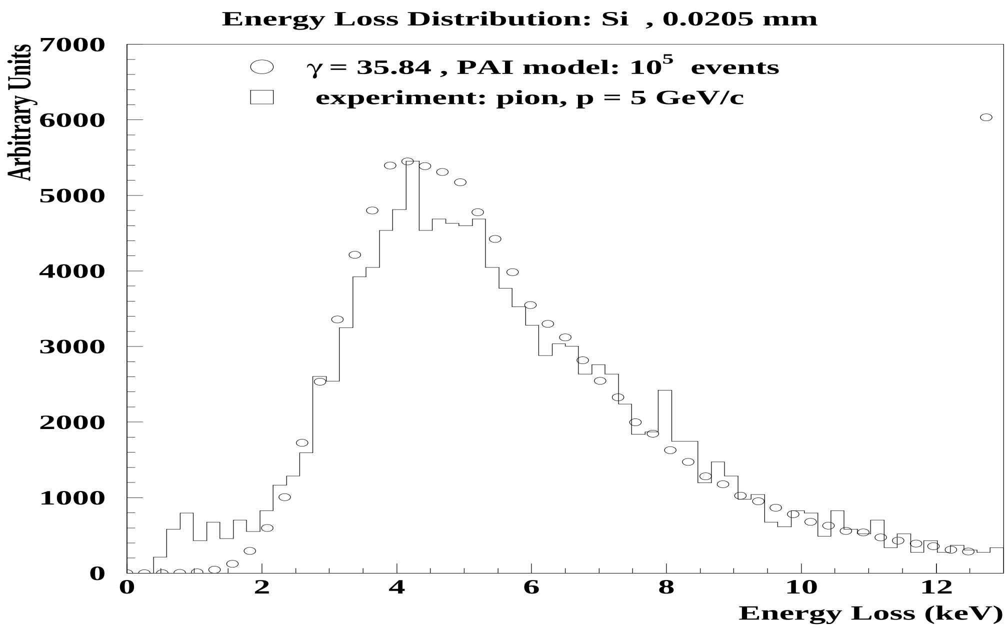

Figure 3: The ionisation energy loss distribution produced by pions with the momentum of 5 GeV/c in silicon with the thickness of 20.5 μm . Histogram is the experimental data , open circles are simulation according to the PAI model.

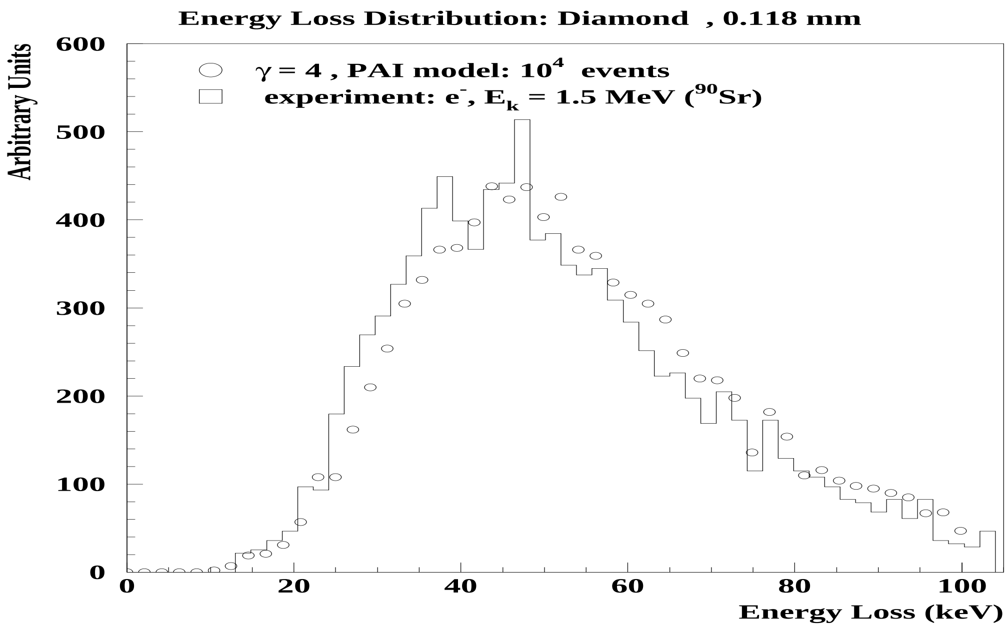

Figure 4: The ionisation energy loss distribution produced by electrons with the momentum of ~ 1.5 GeV/c (^{90}Sr) in polycrystalline diamond with the thickness of 118 μm. Histogram is the experimental data , open circles are simulation according to the PAI model with the Gamma-distributed diamond thickness.

5 General Consideration of Excitation Fluctuations

The energy loss Δ on medium excitations produced by relativistic charged particle crossing a radiator with a fixed length l experiences fluctuations due to both the number of emitted excitations n (discrete Poisson distribution) and their energies ω (rather than frequencies to avoid extra $\hbar$ in relations):

$$\Delta = \omega_1 + \omega_2 + \cdots + \omega_n. \tag{1}$$

The energy loss distribution $\varphi(\Delta)$ normalized so that the probability to loose the energy in the interval $(\Delta, \Delta + d\Delta)$ is $\varphi(\Delta)\, d\Delta$:

$$\int_0^\infty \varphi(\Delta)\, d\Delta = 1$$

from the mathematical point of view is the modified Poisson distribution describing the fluctuations of the sum of random variables, when the number of these variables fluctuates according to the usual discrete Poisson distribution.

Let us introduce the probability density, $P(x, \omega)$, of the excitation emission with the energy ω at the point x (x is the distance along the trajectory), which is normalized according the following expression:

$$\int_0^l dx \int_{\omega_{min}}^{\omega_{max}} d\omega P(x, \omega) = 1.$$

Select some trajectory with fixed both the number of collisions and the energy transfers in each collision. Obviously, if the excitation number n and the excitation energies ω_k are fixed, then the total energy loss is fixed as well. Therefore in this case the distribution is reduced to the Dirac delta function:

$$\varphi(\Delta) = \delta\left(\Delta - \sum_{k=1}^{n} \omega_k\right).$$

It is usually assumed that the number of excitations emitted from the trajectory with the fixed length fluctuates according to the Poisson distribution.

Then the following expression is valid:

$$\varphi(\Delta) = \sum_{n=0}^{\infty} \left\{ \frac{\bar{N}^n \exp(-\bar{N})}{n!} \prod_{k=1}^{n} \left[\int_0^l dx_k \int_{\omega_{min}}^{\omega_{max}} d\omega_k P(x_k, \omega_k) \right] \delta \left(\Delta - \sum_{k=1}^{n} \omega_k \right) \right\},$$

where $\bar{N}$ is the mean number of the excitations produced along a trajectory of the length l. The energy loss resulting in medium excitations is a non-negative value ($\Delta \geq 0$). Therefore it is convenient to express the delta function with help of the Laplace transformation:

$$\delta \left(\Delta - \sum_{k=1}^{n} \omega_k \right) = \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i} \exp(p\Delta) \prod_{k=1}^{n} \exp(-p\omega_k), \quad Re(p) > 0.$$

Taking into account the identity of integrals on the variables x_k and ω_k, the summation on n in (5) results in the exponential function:

$$\varphi(\Delta) = \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i} \exp(p\Delta - \bar{N}) S(p),$$

$$S(p) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\bar{N} \int_0^l dx \int_{\omega_{min}}^{\omega_{max}} d\omega P(x,\omega) \exp(-p\omega) \right]^n =$$

$$= \exp \left[\int_0^l dx \int_{\omega_{min}}^{\omega_{max}} d\omega R(x,\omega) \exp(-p\omega) \right],$$

where we introduced, $R(x,\omega) = \bar{N} P(x,\omega)$, the local (at the point x) spectral density of the mean number of emitted excitations:

$$\int_0^l dx \int_{\omega_{min}}^{\omega_{max}} d\omega R(x,\omega) = \bar{N}.$$

We have finally:

$$\varphi(\Delta) = \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i} \exp \left\{ p\Delta - \int_0^l dx \int_{\omega_{min}}^{\omega_{max}} d\omega R(x,\omega) \left[1 - \exp(-p\omega)\right] \right\}.$$

We simplify this general equation for the uniform case when P (and hence R) do not depend on x:

$$\int_0^l dx \int_{\omega_{min}}^{\omega_{max}} d\omega R(x,\omega) = \int_{\omega_{min}}^{\omega_{max}} \frac{d\bar{N}}{d\omega} d\omega = \bar{N}.$$

We can therefore write the well known result:

$$\varphi(\Delta) = \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i} \exp\left\{p\Delta - \int_{\omega_{min}}^{\omega_{max}} \frac{d\bar{N}}{d\omega}\left[1 - \exp(-p\omega)\right] d\omega\right\}, \quad Re(p) > 0,$$

Note that the origin of these fluctuations is the same as of the fluctuations of ionisation energy loss described by L. Landau and the fluctuations of the energy loss on transition radiation considered later by V.A. Chechin and V.K. Ermilova. Similar expressions and some approximations were also derived for fluctuations of energy loss on Cherenkov and synchrotron radiations. L. Landau derived the general solution of the problem by the method of integral equation. Here the energy loss on medium excitations was considered based on a statistical approach, which is more general than the method of integral equation.

6 Distribution of ionisation in very thin absorbers

What we measure in proportional detectors like multi-wire proportional chambers or silicon detectors is a signal proportional to ionisation (the number of electron-ion pairs) produced by an incident particle inside the sensitive volume of the detector. We consider now the problem concerning the distribution of ionisation in very thin absorbers when the ionisation is not far from the unit, $n \gtrsim 1$. The number of electron-ion pairs created in one ionisation collision with the energy transfer ω is distributed around the mean ionisation $\bar{n}$,

$$\bar{n} = \frac{\omega}{W},$$

with the variance,

$$< (n - \bar{n})^2 >= F \frac{\omega}{W},$$

where W is the mean energy required for the creation of one electron-ion pair, and $F < 1$ is the Fano factor.

Usually, for simulation it is sufficient to approximate the Fano distribution $p(n)$, by the simple Gaussian:

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(n-\bar{n})^2}{2\sigma^2}\right\}, \quad \sigma^2 = F\frac{\omega}{W}.$$

The latter distribution can be expressed in terms of energy transfers:

$$p(\omega) = \frac{1}{\sigma'\sqrt{2\pi}} \exp\left\{-\frac{(\omega-\omega')^2}{2\sigma'^2}\right\}, \quad \omega' = \bar{n}W, \quad \sigma'^2 = F\frac{\omega}{W}.$$

Then we can consider the influence of the Fano effect on the ionisation cross section $d\sigma_i/d\omega$ by the following convolution:

$$\frac{d\tilde{\sigma}_i}{d\omega} = \int_{I_1}^{\omega_{max}} \frac{1}{\sigma'\sqrt{2\pi}} \exp\left\{-\frac{(\omega-\omega')^2}{2\sigma'^2}\right\} \frac{d\sigma_i}{d\omega'} d\omega',$$

where I_1 is the first ionisation potential and ω_{max} is the kinematic maximum of the energy transfer. The latter cross section can be used in the general solution for the distribution of the energy loss $\varphi(\Delta)$ in the layer with a fixed thickness l:

$$\varphi(\Delta) = \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i} \exp\left\{p\Delta - lN\int_{I_1}^{\omega_{max}} \frac{d\tilde{\sigma}_i}{d\omega}\left[1-\exp(-p\omega)\right] d\omega\right\},$$

where N is the atomic density and the path of integration corresponds to $\mathrm{Re}(p) > 0$. Since the ionisation is the discrete random value, we consider the energy loss distribution as the sum over a fixed losses $\Delta_n = nW$:

$$\varphi(\Delta) = \sum_{n=0}^{\infty} C_n \delta(\Delta - nW),$$

where the coefficients C_n can be considered as the probabilities to measure the ionisation n in absorber with the thickness l. Our problem is reduced to the determination of the coefficients C_n. We should simplify the distribution, indeed:

$$J(p) = lN \int_{I_1}^{\omega_{max}} \frac{d\tilde{\sigma}_i}{d\omega} \left[1 - \exp(-p\omega)\right] d\omega \simeq \sum_{k=1}^{k_{max}} \bar{N}_k \left[1 - \exp(-kpW)\right],$$

where

$$\bar{N}_k = lNW \left(\frac{d\tilde{\sigma}_i}{d\omega}\right)_{\omega=kW}, \quad k_{max} = \frac{\omega_{max}}{W}.$$

Here $\bar{N}_k$ is the mean number of ionizing collisions resulting in the creation of k electron-ion pairs.

It is convenient to represent the δ-function in terms its Laplace transformation:

$$\delta(\Delta - nW) = \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i} \exp\left\{p(\Delta - nW)\right\}, \quad \mathrm{Re}(p) > 0.$$

Then we have:

$$\sum_{n=0}^{\infty} C_n \exp\{-npW\} = \exp\{-J(p)\}.$$

We apply now the operator

$$\int_{-i\pi/W}^{i\pi/W} dp \, \exp\{kpW\},$$

to the both sides of the above equation and get after simple transformations:

$$C_n = W \int_{-i\pi/W}^{i\pi/W} \frac{dp}{2\pi i} \exp\left\{ npW - \sum_{k=1}^{k_{max}} \bar{N}_k \left[1 - \exp(-kpW)\right] \right\}.$$

It is convenient to make the following transformation: $z = \exp(-pW)$. Then equation for C_n becomes:

$$C_n = \oint_{\tilde{C}} \frac{dz}{2\pi i} \frac{F(z)}{z^{n+1}},$$

where

$$F(z) = \exp\left\{-\sum_{k=1}^{k_{max}} \bar{N}_k(1-z^k)\right\},$$

and the contour $\tilde{C}$ corresponds to counterclockwise integration over small, $|z| < 1$ circle around the origin of z-plane. From the residue theory we then have:

$$C_n = \frac{1}{n!}\left[\frac{d^n F}{dz^n}\right]_{z=0} = \frac{1}{n!} F^{(n)}(0).$$

The latter equation means we have reduced our problem to the calculation of n-derivative of the function $F(z)$ at the origin of the complex plane z.

We note that:

$$F^{(1)} = F \cdot G, \quad G(z) = \sum_{k=1}^{k_{max}} k\bar{N}_k z^{k-1},$$

and

$$F^{(n+1)} = (F \cdot G)^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} F^{(n-k)} G^{(k)}.$$

Since

$$G^{(k)}(0) = (k+1)!\bar{N}_{k+1},$$

we have finally the following efficient recurrence relation $n > 0$:

$$C_{n+1} = \sum_{k=0}^{n} \frac{k+1}{n+1} \bar{N}_{k+1} C_{n-k}, \quad C_o = \exp\left\{-\sum_{k=1}^{k_{max}} \bar{N}_k\right\}.$$

Here we can obviously represent our relations in terms of experimentally measured values.

Indeed we have:

$$\sum_{k=1}^{k_{max}} \bar{N}_k = lN \int_{I_1}^{\omega_{max}} \frac{d\tilde{\sigma}_i}{d\omega} d\omega = lN\sigma_i = ln_1 = \bar{N}_o,$$

where n_1 is the specific primary ionisation, and $\bar{N}_o$ is the mean number of ionizing collisions in layer with the thickness l. Relation for C_{n+1} modifies the recurrence relation for the usual Poisson distribution ($\bar{N}_{k>1} = 0$ or $\bar{N}_o = \bar{N}_1$):

$$C_{n+1} = \frac{\bar{N}_o}{n+1} C_n, \quad C_o = \exp(-\bar{N}_o) \rightarrow C_n = \frac{\bar{N}_o^n}{n!} \exp(-\bar{N}_o).$$

Thus recurrence relation for C_{n+1} is valid for any modified Poisson distribution, i.e. thedistribution of the sum of discrete random variables when the number of variables is distributed according to the usual Poisson relation. For numerical calculations, relation for C_{n+1} is efficient when $\bar{N}_o \lesssim 100$ or for gas layers $lP \lesssim 3\,\mathrm{cm} \cdot \mathrm{atm}$, otherwise C_o becomes too small and recurrence relation accumulates the computer precision errors.