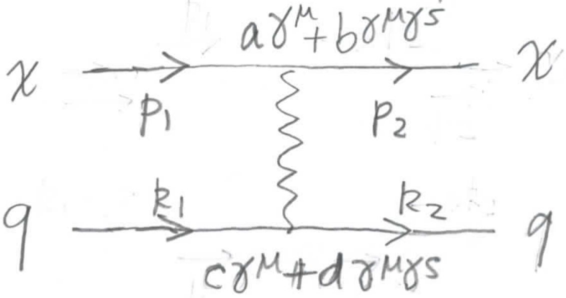
Exercise 9 Spin-Independent and Spin-Dependent non-relativistic scattering.

Prove that the (t-channel, as shown in the diagram) exchange of a vector boson V_μ with coupling

$$\mathcal{L} \ni V_\mu \left[\bar{\psi}_\chi (a\gamma^\mu + b\,\gamma^\mu\gamma^5)\psi_\chi + \bar{\psi}_q (c\gamma^\mu + d\,\gamma^\mu\gamma^5)\psi_q \right]$$

with a dark mater Dirac fermion χ and a quark q (also a Dirac fermion, i.e.



$\chi q \longrightarrow \chi q$

(p_1, p_2, k_1, k_2 are four-momenta)

leads to an interaction amplitude in the non-relativistic limit

$$\mathcal{M} \sim ac + bd\, \vec{\sigma}_\chi \cdot \vec{\sigma}_q$$

where the 1st term corresponds to a spin-independent interaction and the 2nd to a spin dependent interaction ($\vec{\sigma}$ are Pauli matrices)

Help - For those of you who work in astrophysics here are some explanation you may need. The elastic scattering cross section $\sigma = \text{const.} |\mathcal{M}|^2$ (const. is a constant depending of the masses). $\mathcal{M}$, the invariant amplitude is defined in this case as

Exercise 9 (continuation)

$$\mathcal{M} = K\ \bar{u}_{\chi}(p_2)\,[a\gamma^{\mu} + b\gamma^{\mu}\gamma^5]\,u_{\chi}(p_1)\ \ \bar{u}_q(k_2)\,[c\gamma^{\mu} + d\gamma^{\mu}\gamma^5]\,u_q(k_1)$$

(another constant)

In Dirac's representation, the γ matrices are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the Dirac spinors:

$$\boxed{u^{1(2)} = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} \phi^{1(2)} \\ \dfrac{c(\vec{\sigma}\cdot\vec{p})}{E+mc^2}\,\phi^{1(2)} \end{pmatrix}}$$

where $\phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices

The non-relativistic limit $\vec{v} \to 0$ simplifies the calculations considerably: just put to zero any velocity dependent term.

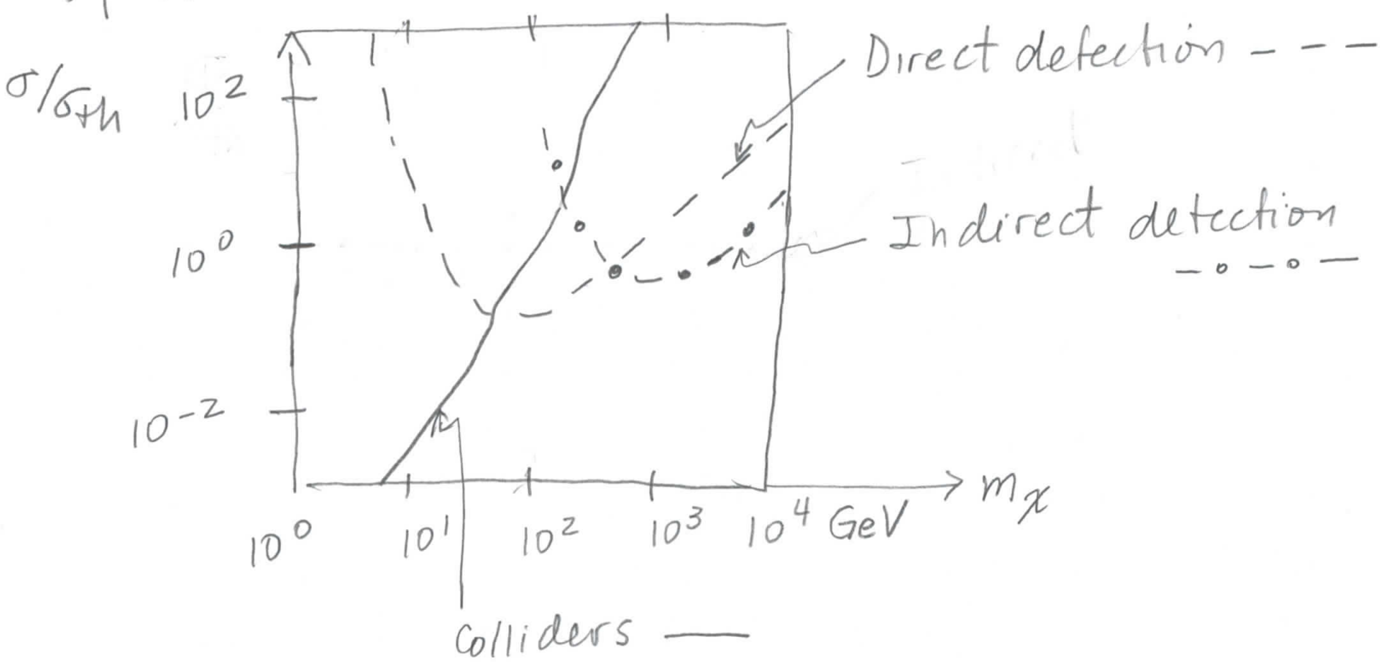
Nonrelativistic limit: $\begin{cases} E = mc^2 + \dfrac{mv^2}{2} \longrightarrow mc^2 \\ \vec{p} = m\vec{v} \longrightarrow 0 \end{cases}$

One more definition $\bar{u} \equiv u^{+}\gamma^0$ and $u^{+} = (u^*)^T$

u^*: complex conjugate of u, u^T = transposed of u

Exercise 10 (the last one)

Consider the combined limits on a particular model represented schematically in the plot derived under the assumption that the particular DM particle accounts for the whole of the DM (simplified version of limits in 1305.1605)



Show how these limits would change if the dark matter particle χ would instead constitute only a fraction $f = \rho_\chi / \rho_{DM}$ of the DM, say $f = 10^{-1}$.

(this exercise is very simple - do not look for a complicated argument)

Here σ_{th} is a constant (reference) cross section.